CHAPTER 6

# Modeling with Discrete Dynamical Systems

## 6.1 INTRODUCTION

One of the most exciting areas of modeling concerns predicting temporal evolution. The main question that is posed in this setting is how do variables of interest change over time? This type of problem is everywhere to be found, for example in areas as diverse as science, engineering and finance. Prediction means that given the values of the variables at a certain instant of time we can predict, i.e. compute their values at any future time. A system of equations that allows such a prediction is called a *Dynamical System*.

In this chapter we consider discrete dynamical systems. The mathematical assumption is that the time variable $n$ is incremented discretely and corresponds to the integers $\{0, 1, 2, 3, 4, \dots\}$. The value of a variable $x$ of interest is then a sequence $\{x_0, x_1, x_2, x_3, x_4, \dots\}$. Now the problem of modeling is to determine an equation of the form

$$x_{n+1} = x_n + \Delta x_n$$

and this is done by estimating how the variable $x_n$ changes as $n$ is incremented from time $n$ to time $n + 1$.

We develop this topic along the following four complementary lines:

- numerical solutions,
- analytical solutions,
- qualitative behavior,
- modeling techniques.

As the terminolgy suggests, numerical approaches to difference equations will involve direct computation of these sequences via computer. In contrast, analytical solution methods seek closed form solutions; these are available only in limited circumstances.

Qualitative approaches are analytical as well as numerical approaches to determine the qualitative behaviour of the solutions in the long run. The questions addressed are: do the solutions go off to infinity, do they approach a finite value, will they oscillate or behave more complicated? Another question of interest is the sensitivity of solutions to variation of parameters. A change in the qualitative behaviour when a parameter is varied is called a bifurcation.

The topic of modeling will treat empirical and qualitative approaches for constructing difference equations. We will consider the development of models

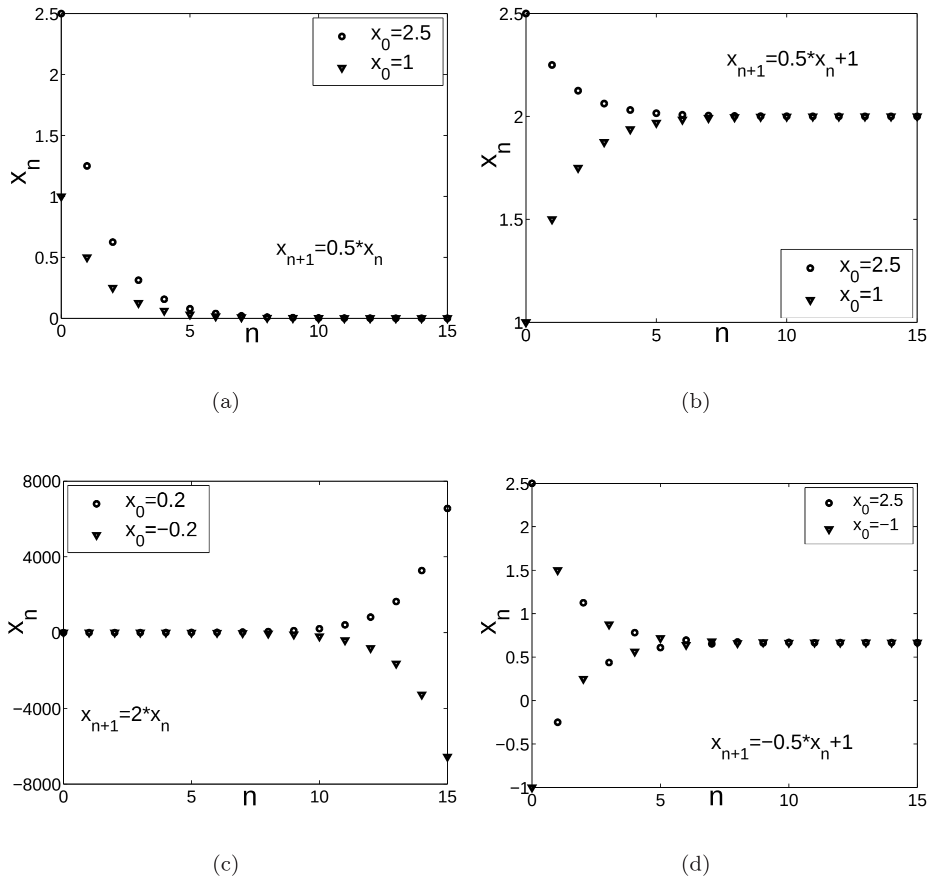


**FIGURE 6.1:** Comparison of the numerical solutions for some simple difference equations.

based on the qualitative approaches presented in Chapter 2 as well as the more quantitative data fitting approaches of Chapter 5.

A simple but nevertheless important difference equation is the equation

$$x_{n+1} = ax_n + b. \tag{6.1}$$

If an initial value $x_0$ is fixed the solution is determined for all $n$,

$$x_1 = ax_0 + b, \;\; x_2 = ax_1 + b, \;\; x_3 = ax_2 + b, \;\; \ldots.$$

Numerically simulated solutions of (6.1) for various values of the parameters $a$ and $b$ are shown in Figure 6.1. In Figure 6.1 (a) we see that the solutions decay to zero while in Figure 6.1 (b) they tend to the value 2. In Figure 6.1 (c) the initial values are close to zero. Both solutions remain close to zero for a while, but eventually they split apart and tend to $\pm\infty$. In Figure 6.1 (d) the solutions tend to $x \approx 0.7$. Here the solutions alternate between values above and below 0.7 when approaching this value. Thus, a noticeable feature for all of these solutions is the long term behavior. Qualitatively we say the solution either blows up or approaches a finite limiting value.

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**EXAMPLE 6.1 Discrete Compound of Interest**

Interest rates for loans or saving accounts are normally fixed on an annual basis, however the compounding scheme typically applies the interest charges monthly. Suppose you purchase something for a certain amount of $\$a_0$ and charge it to your credit card that carries an annual interest rate of $r\%$. Let $a_n$ be the accumulated debt after $n$ months. In Section 6.2.2 we will see that $a_n$ satisfies the difference equation

$$a_{n+1} = (1 + \frac{r}{1200})a_n - p, \tag{6.2}$$

where $p$ is your monthly payment. Equation (6.2) has the form of Equation (6.1). By solving this equation you can answer questions such as: when is a loan $a_0$ paid off given a certain monthly payment $p$, or what should the monthly payment be in order that the loan is paid off after a prescribed amount of time?

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Equation (6.1) is called a *linear* first order difference equation. It is linear because the right hand side is a linear function of $x_n$. It is of first order because only one time step is involved. The simplest *nonlinear* first order difference equation is

$$x_{n+1} = ax_n + bx_n^2. \tag{6.3}$$

In Figure 6.2 numerical solutions of (6.3) are shown for $b = -1$ and two different values of $a$. In Figure 6.2 (a) we see approach to a limiting value as in Figure 6.1 (d). In contrast in Figure 6.2 (b) the solution eventually alternates between the values 1.6 and 2.7. This type of behavior cannot be found in solutions of linear equations. The solutions of nonlinear equations show a much richer variety of behaviors. Another important difference is that linear equations admit closed form solutions whereas nonlinear equations typically cannot be solved analytically.

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**EXAMPLE 6.2 Population Growth**

Discrete dynamical systems are widely used in population modeling, in particular for species which have no overlap between successive generations and for which births occur in regular, well-defined 'breeding seasons'. Let $p_n$ be the average population of a species between times $n\tau$ and $(n+1)\tau$. The time step $\tau$ depends on the particular species and can range from an hour to several years. For example many species of bamboo grow vegetatively for 20 years before flowering and then dying.

In population dynamics one constructs a model for the change $\Delta p_n = p_{n+1} - p_n$. The simplest model is a linear model, $\Delta p = kp_n + \beta$, where $k$ is called the reproduction rate and $\beta$ models constant immigration ($\beta > 0$) or emigration ($\beta < 0$). The difference equation that results from this model assumption,

$$p_{n+1} - p_n = kp_n + \beta,$$

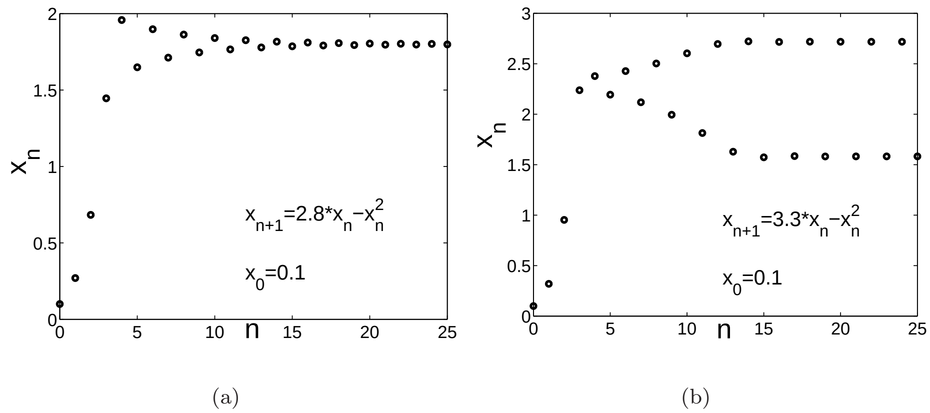


**FIGURE 6.2:** Numerical solutions for equation (6.4).

is again of the form of Equation (6.1).

Competition for resources usually leads to nonlinear difference equations. We will see that the simplest model that takes competition into account leads to the equation

$$p_{n+1} = rp_n - p_n^2, \tag{6.4}$$

which is of the form of Equation (6.3). Equation (6.4) is known in the literature as *logistic map*. Its prominent feature are very complicated, so called chaotic solutions in certain ranges of the parameter $r$.

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The equation

$$x_{n+2} + 2x_{n+1} + 3x_n = \cos(n)$$

is an example of a linear second order difference equation. We shall see that this type of equation always can be transformed to a linear system of two first order equations. The general form of such a system is

$$\begin{aligned} x_{n+1} &= ax_n + by_n + f_n, \\ y_{n+1} &= cx_n + dy_n + g_n, \end{aligned}$$

where $f_n, g_n$ are known sequences. If the right hand sides are replaced by nonlinear functions we have a nonlinear system, for instance

$$\begin{aligned} x_{n+1} &= ax_n - bx_n^2 - cx_ny_n, \\ y_{n+1} &= dx_n - ey_n^2 - fx_ny_n. \end{aligned}$$

This system is used in population modeling as a model for the population growth of two interacting species. The terms $-bx_n^2$ and $-ey_n^2$ model competition within each of the two species whereas the terms $-cx_ny_n$ and $-fx_ny_n$ model competition between the species.

## 6.2 LINEAR FIRST ORDER DIFFERENCE EQUATIONS

### 6.2.1 Analytical Solutions

Possibly the simplest nontrivial difference equation has the form

$$x_{n+1} = ax_n. \tag{6.5}$$

This equation has the special solution $x_n = 0$. Since it is constant it is said to be an equilibrium solution. The value of the constant, $\overline{x} = 0$, is called an equilibrium value or shortly an equilibrium. The solutions for initial values $x_0 \neq 0$ are found by implementing the iteration,

$$\begin{aligned} x_1 &= ax_0 \\ x_2 &= ax_1 = a^2x_0 \\ x_3 &= ax_2 = a^3x_0 \\ &\vdots \\ x_n &= a^nx_0. \end{aligned} \tag{6.6}$$

From (6.6) we can easily infer how the qualitative behavior of $x_n$ depends on $a$: if $|a| > 1$ then $x_n$ goes off to infinity (the equilibrium is said to be unstable), whereas if $|a| < 1$ then $x_n$ tends to 0 (the equilibrium is said to be stable). This explains the behavior of the numerical solutions of Figures 6.1 (a) and (c). Note also that if $a > 0$ then $x_n$ has the same sign as $x_0$ for all $n$. In contrast if $a < 0$ the solution alternates between positive and negative values.

The cases $a = 1$ and $a = -1$ are special. If $a = 1$ we have $x_n = x_0$ for all $n$, hence every $x$ is an equilibrium. If $a = -1$ the solution $x_n = (-1)^nx_0$ flips back and forth between $x_0$ and $-x_0$.

A more general equation is the following,

$$x_{n+1} = ax_n + b. \tag{6.7}$$

An equilibrium is determined by $x_{n+1} = x_n = \overline{x}$ for all $n$, hence

$$\overline{x} = a\overline{x} + b \quad \Rightarrow \quad \overline{x} = \frac{b}{1-a},$$

where we assume $a \neq 1$. We can transform (6.7) to (6.5) by subtracting the equilibrium. Set

$$y_n = x_n - \overline{x}.$$

Then

$$\begin{aligned} y_{n+1} &= x_{n+1} - \overline{x} = ax_n + b - \overline{x} \\ &= a(y_n + \overline{x}) + b - \overline{x} = ay_n, \end{aligned}$$

and so $y_n = a^ny_0$. The solution of (6.7) is found by transforming $y_n$ back to $x_n$,

$$x_n = y_n + \overline{x} = ax^n(x_0 - \overline{x}) + \overline{x} = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}. \tag{6.8}$$

Again the value of $|a|$ determines whether $x_n$ goes off to infinity or approaches $\overline{x}$, and the sign of $a$ determines whether $x_n - \overline{x}$ alternates or has a constant sign.

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### EXAMPLE 6.3

The equation

$$x_{n+1} = \frac{1}{2}x_n + 1$$

is of the form of (6.7). The equilibrium is

$$\overline{x} = \frac{b}{1-a} = 2.$$

Since $a = 1/2 < 1$ and $a > 0$ the solutions approach the equilibrium 2 and the sign of $x_n - 2$ is the same for all $n$. This explains the behavior of the numerical solutions shown in Figure 6.1 (d).

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A more general form than (6.7) is provided by the equation

$$x_{n+1} = ax_n + b_n, \tag{6.9}$$

where $b_n$ is a given sequence. This equation is said to be nonhomogeneous due to the presence of the $b_n$ term. If $b_n = 0$ for all $n$, (6.9) simplifies to (6.5) and then the equation is called homogeneous. We refer to (6.5) as the homogeneous equation associated with (6.9). In the special case in which $b_n = b =$ const we were able to transform the nonhomogeneous equation to its associated homogeneous equation, but if $b_n$ varies with $n$ this is no longer possible.

Definition **4**. A one parameter family of solutions of (6.9) is an expression $x_n = x_n(c)$ that depends linearly on a parameter $c$ and satisfies (6.9) identically in $n$ and $c$. A particular solution is a solution that contains no free parameters. A one parameter family of solutions is a general solution if for every particular solution $p_n$ we can find a value $\overline{c}$ of $c$ such that $p_n = x_n(\overline{c})$ for all $n$.

Consider now the *difference* $h_n = q_n - p_n$ of two particular solutions $q_n$ and $p_n$ of (6.9). The computation

$$\begin{aligned} h_{n+1} &= q_{n+1} - p_{n+1} = (aq_n + b_n) - (ap_n + b_n) \\ &= a(q_n - p_n) = ah_n \end{aligned}$$

shows that $h_n$ is a solution of the homogeneous equation (6.5). Since $h_0 = q_0 - p_0$ it follows from (6.6) that $h_n = (q_0 - p_0)a^n$ and so,

$$q_n = (q_0 - p_0)a^n + p_n.$$

If we assume $p_n$ is a known particular solution, this equation allows to find any other particular solution $q_n$ from its initial value $q_0$. Thus if we write

$$x_n = ca^n + p_n, \tag{6.10}$$

and consider $c$ as parameter, the solution $q_n$ is simply obtained by setting $c = q_0 - p_0$. We therefore have proved the following theorem.

THEOREM **5**. Let $p_n$ be a particular solution of the nonhomogeneous equation

$$x_{n+1} = ax_n + b_n.$$

Then the family

$$x_n = ca^n + b_n$$

is a general solution.

Note that there is no unique general solution. For instance,

$$x_n = ca^n + (p_n + 5a^n)$$

is also a general solution because $p_n + 5a^n$ is another particular solution.

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### EXAMPLE 6.4

Verify that $p_n = -n - 1$ is a particular solution of

$$x_{n+1} = 3x_n + 2n + 1.$$

***Solution*** To test that an expression is a solution of a difference equation we just have to plug it into the equation and check if both sides are the same. Now the left hand side evaluates to

$$p_{n+1} = -(n + 1) - 1 = -n - 2,$$

and the right hand side to

$$3p_n + 2n + 1 = 3(-n - 1) + 2n + 1 = -n - 2.$$

Since these are the same we have verified that $p_n$ is a solution. It is a particular solution because it does not depend on parameters.

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### EXAMPLE 6.5

Find the general solution of

$$x_{n+1} = 3x_n + 2n + 1$$

and the particular solution that satisfies $x_0 = 1$.

***Solution*** Form Example 6.4 we know that $p_n = -n - 1$ is a particular solution. Since $a = 3$ the general solution is

$$x_n = c3^n - n - 1.$$

To find the particular solution asked for we evaluate at $n = 0$,

$$x_0 = c - 1 = 1.$$

It follows that $c = 2$, hence

$$x_n = 2\,3^n - n - 1$$

is the solution with $x_0 = 1$.

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| $b_n$ | form of particular solution | conditions |
|---|---|---|
| (6.11) | $p_n = (A_0 + A_1 n + \cdots + A_M n^M) b^n$ | $b \neq a$ |
| | $p_n = n(A_0 + A_1 n + \cdots + A_M n^M) b^n$ | $b = a$ |
| (6.12) | $p_n = (A_0 + A_1 n + \cdots + A_M n^M) b^n \cos(kn)$ $+ (B_0 + B_1 n + \cdots + B_M n^M) b^n \sin(kn)$ | $k \neq 0, \pi$ |

**TABLE 6.1:** Solution forms $p_n$ for $b_n$ given by Equations (6.11) and (6.12)

To complete the solution of the nonhomogeneous equation (6.9) we need to find a particular solution. For general terms $b_n$ this can be a complicated task. However there is a method that applies always if $b_n$ is a combination of powers of $n$ ($n^0, n^1, n^2$ etc.), trigonometric functions of $n$, and powers $b^n$. This method is called method of undetermined coefficients.

**Method of undetermined coefficients.** Assume $b_n$ has one of the following forms,

$$b_n = (c_0 + c_1 n + \ldots + c_M n^M) b^n, \tag{6.11}$$

where $c_M \neq 0$, or

$$\begin{aligned} b_n &= (c_0 + c_1 n + \cdots + c_M n^M) b^n \cos(kn) \\ &+ (d_0 + d_1 n + \ldots + d_M n^M) b^n \sin(kn), \end{aligned} \tag{6.12}$$

where at least one of $c_M$ or $d_M$ is nonzero. The coefficients $b$, $k$ and $c_j, d_j$ $(0 \leq j \leq M)$ are assumed to be given numbers. It can be shown that if $b_n$ is as in (6.11) or (6.12), then there exists a unique particular solution $p_n$ of the form as summarized in Table 6.2.1. To find the values of the coefficients $A_j, B_j$ $(0 \leq j \leq M)$, one sets up a trial form for $p_n$ according to the table with initially undetermined values of the coefficients, substitutes the trial form into the difference equation, and determines the values of the coefficients from the condition that $p_n$ be a solution. If $b_n$ is a linear combination of several terms of the form of (6.11) or (6.12), with different values of $b$ or $(b, k)$, each of them can be treated separately and the results are added up.

## EXAMPLE 6.6

Find a particular solution of

$$x_{n+1} = 3x_n + 2n + 1.$$

***Solution*** Here $b_n = 2n + 1$ is of the form (6.11) with $b = 1$ and $M = 1$. Thus we use $p_n = A + Bn$ as trial form and substitute this into the difference equation to obtain,

$$A + B(n+1) = 3(A + Bn) + 2n + 1,$$

or

$$(2A - B + 1) + (2B + 2)n = 0.$$

This equation holds for all $n$ if $A$ and $B$ satisfy the equations $2A - B = -1$ and $2B = -2$. The solution is $A = B = -1$, hence

$$p_n = -n - 1.$$

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### EXAMPLE 6.7

Find a particular solution of

$$x_{n+1} = -x_n + \cos 2n.$$

***Solution*** Substitution of the trial form $p_n = A\cos 2n + B\sin 2n$ into the difference equation yields

$$A\cos 2(n+1) + B\sin 2(n+1) = -A\cos 2n - B\sin 2n + \cos 2n.$$

We apply the formulae for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ to the terms on the left hand side and then rearrange the equation as

$$[A(1+\cos 2) + B\sin 2 - 1]\cos 2n + [-A\sin 2 + B(1+\cos 2)]\sin 2n = 0.$$

This equation holds for all $n$ if the terms in both brackets vanish. Setting these terms equal to zero gives the following system of equations for $A$ and $B$,

$$\begin{aligned} A(1+\cos 2) + B\sin 2 - 1 &= 0 \\ -A\sin 2 + B(1+\cos 2) &= 0, \end{aligned}$$

with the solution

$$A = \frac{1}{2}, \quad B = \frac{\sin 2}{2(1+\cos 2)}.$$

Hence the particular solution is

$$p_n = \frac{1}{2}\cos 2n + \frac{\sin 2\cos 2n}{2(1+\cos 2)} = \frac{\cos 2n + \cos 2(n-1)}{2(1+\cos 2)}.$$

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### EXAMPLE 6.8

Find a particular solution of

$$x_{n+1} = x_n/2 + n(1/2)^n.$$

***Solution*** Here $a = b = 1/2$, so the trial function is $p_n = n(An+B)(1/2)^n$. Again we substitute $p_n$ into the difference equation,

$$[A(n+1)^2 + B(n+1)](1/2)^{n+1} = (An^2 + Bn)(1/2)^n/2 + n(1/2)^n.$$

We multiply this equation by $2^{n+1}$ and rearrange terms as

$$2(A-1)n + (A+B) = 0.$$

Thus $A = -B = 1$ and the particular solution is

$$p_n = (n^2 - n)(1/2)^n.$$

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## 6.2.2 Modeling Examples

### (A) Savings Accounts and Loans

**Savings Accounts.** Assume you open a savings account at an annual interest rate of $r\%$ and with monthly compound of interest. Let $a_n$ be the dollar amount on the account at the end of month $n$ after the opening date. The amount at the end of month $n+1$ is

$$a_{n+1} = a_n + i_n + p_n,$$

where $p_n$ is the total deposit (withdrawal if $p_n < 0$) and $i_n$ is the interest earned,

$$i_n = \left(\frac{r}{100}\frac{1}{12}\right)a_n.$$

Thus $a_n$ satisfies the nonhomogeneous, linear first order difference equation,

$$a_{n+1} = ka_n + p_n, \tag{6.13}$$

where

$$k = 1 + \frac{r}{1200}.$$

If $p_n = p = const$ we know the solution already (Equation (6.8) with $a = k$, $b = p$, $x_n = a_n$),

$$a_n = k^n(a_0 + \frac{p}{k-1}) - \frac{p}{k-1} = k^n a_0 + \frac{(k^n-1)p}{k-1}. \tag{6.14}$$

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**EXAMPLE 6.9**

After graduating from High School Peter works for four years. During this time he deposits each month \$1000 on a savings account at an annual interest rate of 5% (no initial deposit). The next four years Peter spends on College. During this time he withdraws each month an amount of $\$p_w$ from his savings account so that at the end of the four years the balance is zero again. Find $p_w$ and the total interest earned.

***Solution*** Letting $p$ be the the monthly deposit, the accumulated amount on Peter's savings account after the first four years is

$$a_{48} = \frac{(k^{48}-1)p}{k-1}.$$

After the second four years this has evolved into

$$a_{96} = k^{48}a_{48} - \frac{(k^{48}-1)p_w}{k-1} = \frac{k^{48}-1}{k-1}(k^{48}p - p_w).$$

Solving the equation $a_{96} = 0$ for $p_w$ gives $p_w = k^{48}p$. With $p = \$1000$ and $k = 1+5/1200$ this evaluates to $p_w = \$1220.89$. The total interest earned is $48(p_w - p) = \$10,602.72$.

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**Loans.** Equation (6.13) also holds for loans. In this case $a_0$ is the amount borrowed and $a_n$ is the amount owed after $n$ months. The term $-p_n > 0$ is the monthly payment. For constant monthly payment $p$ the difference equation for $a_n$ is

$$a_{n+1} = ka_n - p, \tag{6.15}$$

with the solution

$$a_n = ka_0^n - \frac{(k^n-1)p}{k-1}.$$

Note that (6.15) has an unstable equilibrium $\overline{a} = p/(k-1)$. If $a_0 > \overline{a}$ the solution grows without bound when $n$ increases. While for savings accounts this may be desirable, it is certainly not tolerable for loans.

The term of a loan is the time $N$ (in months) when the loan is paid off. Setting $a_N = 0$ leads to a linear relation between monthly payment and initial debt,

$$p = \frac{k^N(k-1)}{k^N-1}a_0. \tag{6.16}$$

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**EXAMPLE 6.10**

You decide to purchase a home with a mortgage at 6% annual interest and with a term of 30 years. For $k = 1 + 6/1200 = 1.005$ and $N = 360$ the factor

$$R = \frac{k^N(k-1)}{k^N-1}$$

in Equation (6.16) is $R = 0.00600$. If the house costs $a_0 = \$200,000$, the monthly payment is $p = Ra_0 = \$1,199.10$. On the other hand, if your income restricts your monthly payment to a maximum of $p_m = \$1000$, the maximal amount you can spend for the house is $p_m/R = \$166,791.61$.

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If $p$, $a_0$ and $k$ are fixed, the equation (6.16) may be considered as an equation for the term $N$. Writing $k^N = e^{N \ln k}$, Equation (6.16) can be rewritten as

$$e^{N \ln k} = \frac{p}{p-(k-1)a_0},$$

hence

$$N = \frac{-\ln[1-(k-1)a_0/p]}{\ln k}. \tag{6.17}$$

Note however that the right hand side of (6.17) needs not to be an integer. Nevertheless it can be used to estimate $N$ and then to improve $p$ or $a_0$. For example, assume you need \$200,000 and you want your payment to be close to, but not above \$1500. With $r = 8\%$, $a_0 = 200,000$ and $p = 1500$, (6.17) evaluates to $N = 330.68$. If this is rounded up to $N = 331$, Equation (6.16) gives $p = \$1499.60$.

In our last example on savings accounts and loans we have to solve the nonhomogeneous equation (6.9) with nonconstant $b_n$.

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## EXAMPLE 6.11

An employee starts her position at the age of 25 with an annual salary of \$40,000. She deposits each month 8% of her monthly salary on a retirement savings account. The salary increases by 3% each year and the annual interest rate of her retirement savings account is 6%. What is the accumulated amount when she retires at the age of 65?

***Solution*** Let $A_m$ be the accumulated amount on the retirement savings account at the end of year $m$ and let $a_{m,n}$ be the accumulated amount in month $n$ of year $m+1$, that is,

$$a_{m,0} = A_m, \quad a_{m,12} = A_{m+1}.$$

The amount $a_{m,n}$ satisfies difference equation

$$a_{m,n+1} = k_r a_{m,n} + k_p s_m, \tag{6.18}$$

where $k_r = 1 + 6/1200 = 1.005$, $k_p = 8/1200$ and $s_m$ is the salary in year $m+1$. The salary satisfies the homogeneous difference equation

$$s_{m+1} = k_s s_m,$$

with $k_s = 1 + 3/100 = 1.03$ and $s_0 = 40,000$, hence

$$s_m = k_s^m s_0.$$

The solution of (6.18) is

$$a_{m,n} = k_r^n a_{m,0} + \frac{(k_r^n - 1)k_s^m k_p s_0}{k_r - 1}.$$

Evaluating this at $n = 12$ yields

$$A_{m+1} = k_a A_m + f k_s^m, \tag{6.19}$$

where

$$f = \frac{(k_r^{12} - 1)k_p s_0}{k_r - 1} = \$3289.48, \quad k_a = k_r^{12} = 1.0616778.$$

It remains to solve the nonhomogenous difference equation (6.19). By using the method of undetermined coefficients a particular solution can be determined to be $p_m = f k_s^m/(k_s - k_a)$. The solution with initial value $A_0 = 0$ then is

$$A_m = \frac{(k_s^m - k_a^m)f}{k_s - k_a}.$$

For $m = 40$ this evaluates to $A_{40} = 799,106.39$. Hence the employee starts her retirement with an amount of \$799,106.39 on her retirement savings account.

### (B) Cooling and Heating

Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference of the temperature of the object and its surrounding. Let $\Delta T_n = T_{n+1} - T_n$ be the change in temperature of the object over a time interval $\tau$, typically $\tau = 1$ hour. According to Newton's law of cooling we have

$$\Delta T_n \propto R_n - T_n,$$

or

$$\Delta T_n = k(R_n - T_n),$$

where $R_n$ is the surrounding temperature. Since we know that temperature decreases if $R_n > T_n$ it follows that $k > 0$. The difference equation that arises from this model is

$$T_{n+1} = T_n + k(R_n - T_n).$$

If $R_n = R =$ const this equation is again of the form (6.7) with solution

$$T_n = (1 - k)^n (T_0 - R) + R.$$

Note that the equilibrium solution is $T_n = R$ as expected. The equilibrium is stable if $0 < k < 2$. However if $1 < k < 2$ the temperature would oscillate about the surrounding temperature which does not make sense physically, hence $0 < k < 1$.

### EXAMPLE 6.12

A murder victim is discovered in an office building that is maintained at 68 degrees F. Given the medical examiner found the body temperature to be 88 degrees F at 8am and that one hour later the body temperature was 86 degrees F, at what time was the crime committed?

***Solution*** Setting $T_0 = 98.6$ (where 0 is the time the crime was committed) and $R = 68$ we obtain

$$T_n = 68 + 30.6(1 - k)^n.$$

If we define $n_1$ as the time the body was observed initially by the medical examiner and the time one hour later as $n_1 + 1$ we have the equations

$$T_{n_1} = 88 = 68 + 30.6(1-k)^{n_1},$$

and

$$T_{n_1+1} = 86 = 68 + 30.6(1-k)^{n_1+1}.$$

These two equations may be solved to give $k = 1/10$ and $n_1 = 4.036$. So the crime was committed just before 4am.

## 6.3 LINEAR SECOND ORDER EQUATIONS

### 6.3.1 Homogeneous Equations

We begin by considering the second order linear homogeneous difference equation

$$x_{n+2} + \alpha x_{n+1} + \beta x_n = 0 \tag{6.20}$$

It is readily verified that this equation has solutions of the form

$$x_n = \lambda^n$$

Upon substitution into Equation (6.20) we obtain the *auxiliary* equation

$$\lambda^2 + \alpha\lambda + \beta = 0$$

This quadratic equation has solutions that break down into three cases: i) both solutions real and distinct, ii) one real double solution, and iii) a pair of complex solutions as

$$\lambda_{\pm} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$$

**Case i: $\alpha^2 - 4\beta > 0$. Two real roots.**

In this case

$$\lambda_{+} = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}$$

and

$$\lambda_{-} = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$$

are both real and the solution is

$$h_n = c_1(\lambda_+)^n + c_2(\lambda_-)^n$$

Since the equation is linear we know that the superposition of solutions is again a solution. Notice that there are now two free parameters $c_1$ and $c_2$ to accommodate the two initial conditions $x_0$ and $x_1$ required for a second order difference equation. Notice also that $h_n \to 0$ for $n \to \infty$ if $\lambda_{\pm}| < 1$, but in general $|h_n| \to \infty$ if $|\lambda_+| > 1$.

**EXAMPLE 6.13**

$$x_{n+2} = x_{n+1} + x_n$$

The auxiliary equation is now

$$\lambda^2 - \lambda - 1 = 0$$

The solutions to this quadratic are

$$\lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

Thus, the general solution to the homogeneous problem is

$$h_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

If we select $h_0 = h_1 = 1$ we have the *Fibonocci sequence* $\{1, 1, 2, 3, 5, 8, 13, \dots\}$. Employing this pair of initial conditions it is easily shown that the particular solution is

$$h_n = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

You might impress your friends by telling them the 50th number in this sequence $h_{50} = 20365011074$. It is also apparent that these numbers increase exponentially fast.

**Case ii:** $\alpha^2 - 4\beta = 0$. **One real (double) root.**

In this case

$$\lambda_+ = \lambda_- = -\frac{\alpha}{2}$$

so we only have one solution while we require two for the general solution of a second order difference equation.

It is not hard to verify that in this instance the second solution is actually

$$x_n = n(\frac{-\alpha}{2})^n$$

. (See Exercise 6.15). Now the general solution to this homogeneous equation is

$$h_n = c_1(-\frac{\alpha}{2})^n + c_2 n(-\frac{\alpha}{2})^n$$

**EXAMPLE 6.14**

$$x_{n+2} + 2x_{n+1} + x_n = 0$$

The auxiliary equation is

$$\lambda^2 + 2\lambda + 1$$

which has the solution $\lambda = -1$. Thus, the general solution to this homogeneous problem is

$$h_n = c_1(-1)^n + c_2 n(-1)^n$$

**Case iii: $\alpha^2 - 4\beta < 0$. Two complex roots.**

The solution to the auxiliary equation is again

$$\lambda_\pm = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$$

Based on the fact $\alpha^2 - 4\beta < 0$ we rewrite this as

$$\lambda_\pm = \frac{-\alpha}{2} \pm i\frac{\sqrt{4\beta - \alpha^2}}{2}$$

where $i = \sqrt{-1}$.[1]

We could now write the solution

$$h_n = c_1\left(\frac{-\alpha}{2} + i\frac{\sqrt{4\beta - \alpha^2}}{2}\right)^n + c_1\left(\frac{-\alpha}{2} - i\frac{\sqrt{4\beta - \alpha^2}}{2}\right)^n$$

but this form would not provide much insight. Instead we employ Demoivre's theorem that states

$$\exp(inx) = \cos(nx) + i\sin(nx)$$

To exploit this formula we need to recall that each solution to the auxiliary equation can be written in its complex polar form

$$z = x + iy = r\exp(i\theta)$$

where $x = r\cos\theta$ and $y = r\sin\theta$. Thus, we take

$$x = \frac{-\alpha}{2}, \quad \text{and} \quad y = \frac{\sqrt{4\beta - \alpha^2}}{2}$$

To compute the polar form we need $r$ and $\theta$. Recall

$$r^2 = x^2 + y^2$$

so

$$\begin{aligned} r^2 &= (\frac{-\alpha}{2})^2 + (\frac{\sqrt{4\beta - \alpha^2}}{2})^2 \\ &= \beta \end{aligned}$$

So

$$r = \sqrt{\beta}$$

The angle satisfies

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{4\beta - \alpha^2}}{-\alpha}$$

In polar form, the solution is

$$h_n = c_1 r^n \exp(in\theta) + c_2 r^n \exp(-in\theta)$$

[1] Unlike the previous cases we now assume familiarity with basic complex numbers.

The associated real form is

$$h_n = r^n(c_1 \cos(n\theta) + c_2 \sin(n\theta),$$

where we have used the facts that $\exp(in\theta) = \cos(n\theta) + i\sin(n\theta)$ and that the real and imaginary parts of a complex solution are real solutions (see problems). The form of the solution tells that $h_n \to 0$ for $n \to \infty$ if $r < 1$ and $|h_n| \to \infty$ if $r > 1$. If $r = 1$ the solution remains bounded, but does not approach zero.

---

**EXAMPLE 6.15**

Find the general solution to the homogeneous difference equation

$$x_{n+2} + 2x_{n+1} + 5x_n = 0$$

The auxiliary equation gives the solutions

$$\lambda_{\pm} = -2 \pm i$$

If we write these in polar form we have

$$h_n = 5^{n/2}(c_1 \exp(in\theta) + c_2 \exp(-in\theta))$$

where $\tan\theta = 1/2$. The associated real valued form is

$$h_n = 5^{n/2}(c_1 \cos(n\theta) + c_2 \sin(n\theta)).$$

---

### 6.3.2 The Cobweb Model Revisited

Consider a supply curve

$$p = m_s q + b_s$$

and a demand curve

$$p = m_d q + b_d$$

Here we derive a formula for the values $(q_n, p_n)$ that are the iterations along the supply and demand curves that either converge to an economic equilibrium or spiral out of control. Let the starting point on the demand curve be $(q_0, p_0)$. The next iteration is then given by

$$(q_1, p_1) = (\frac{p_0 - b_s}{m_s}, p_0)$$

Similarly,

$$(q_2, p_2) = (q_1, m_d q_1 + b_d),$$

$$(q_3, p_3) = (\frac{p_2 - b_s}{m_s}, p_2)$$

and

$$(q_4, p_4) = (q_3, m_d q_3 + b_d)$$

Thus, we have established the following pattern:

$$(q_{2n}, p_{2n}) = (q_{2n-1}, m_d q_{2n-1} + b_d)$$

and

$$(q_{2n+1}, p_{2n+1}) = (\frac{p_{2n} - b_s}{m_s}, p_{2n})$$

It is now possible to create a second order difference equation for both $q_n$ and $p_n$. Since

$$q_{2n+1} = \frac{p_{2n} - b_s}{m_s}$$

it follows, upon substituting for $p_{2n}$ that

$$q_{2n+1} = \frac{(m_d q_{2n-1} + b_d) - b_s}{m_s}$$

or,

$$q_{2n+1} = \frac{m_d}{m_s} q_{2n-1} + \frac{b_d - b_s}{m_s}. \tag{6.21}$$

**A Nonhomogeneous Second Order Equation.** The equation (6.21) is of the form

$$q_{2n+1} = \alpha q_{2n-1} + \beta$$

This is a nonhomogeneous second order difference equation whose general solution is, as in the first order case, given by

$$x_n = h_n + p_n,$$

where $h_n$ is the general solution of the associated homogeneous equation and $p_n$ is a particular solution of the nonhomogeneous equation.

The associated homogeneous equation is

$$q_{2n+1} = \alpha q_{2n-1}$$

and has the auxiliary equation

$$\lambda^2 = \alpha$$

so the general solution to the homogeneous problem is

$$h_n = c_1 \alpha^{n/2} + c_2 (-\alpha^{1/2})^n$$

As the nonhomogeneous term is a constant we first search for a particular solution of the form $p_n = A$. This must be an equilibrium solution, if it exists. Solving for $A$ then gives

$$A = \alpha A + \beta$$

or

$$A = \frac{\beta}{1 - \alpha}$$

In terms of the original variables of the supply and demand problem, $\alpha = m_d/m_s$, $\beta = (b_d - b_s)/m_s$, the general solution to the nonhomogeneous equation now becomes

$$q_n = c_1\left(\frac{m_d}{m_s}\right)^{n/2} + c_2\left(-(\frac{m_d}{m_s})^{1/2}\right)^n + \frac{b_d - b_s}{m_s - m_d}$$

It is clear from our previous work that this equation will only converge if

$$|\frac{m_d}{m_s}| < 1$$

Note also that if this condition holds then the quantity supplied converges,

$$q_n \to \frac{b_d - b_s}{m_s - m_d}$$

and approaches the market equilibrium.

## 6.4 NONLINEAR DIFFERENCE EQUATIONS AND SYSTEMS IN POPULATION MODELING

In this section we will consider a sequence of modifications of a population model that characterize the modeling process and illustrate how including or deleting terms in equations can have dramatic effects on the predictive powers of a model.

The simplest model for population growth makes the assumption that there is no competition for resources such as nutrients or habitat. This exponential growth is readily captured by the simple difference equation

$$p_{n+1} - p_n = \Delta p_n = kp_n \tag{6.22}$$

where the growth constant $k > 0$ reflects the rate of reproduction. One would assume that for rabbits this constant would be larger than for elephants. Actual values for $k$ must be determined empirically from the data using a data fitting technique such as least squares.

If instead of simply taking $k > 0$ in Equation (6.22) we could have modeled both the birth rate $k_b$ and the death rate $k_d$ such that

$$p_{n+1} - p_n = k_b p_n - k_d p_n \tag{6.23}$$

Clearly now we may write

$$k = k_b - k_d$$

and as we would expect, if $k > 0$ the model predicts that the population grows exponentially fast and if $-1 < k < 0$ then the population decays exponentially fast. Values of $k$ in the rang $k < -1$ do not make sense because then the solution would oscillate between positive and negative values.

The effect of adding to a population via immigration or subtracting via emigration is captured by

$$p_{n+1} - p_n = kp_n + \beta_n \tag{6.24}$$

where $\beta_n$ is the net flux of population. Now we might expect that growth rates could be offset by immigration or emigration. For example $k < 0$ but $\beta_n = \beta$ can produce a positive equilibrium population.

Obviously ignoring competition for finite resources places significant limitations on this model. It will work well where the assumptions hold true but when the effects of competition for resources become important it will not capture them. To model competition we may argue as follows: competition occurs when there is interaction between two members of a species and the total amount of competition is the number of ways we can select subsets of 2 from a population p which is

$$\text{number of pairwise interactions} \propto \frac{p(p-1)}{2}$$

where we have divided by two to compute the number of combinations rather than permutations. Now we may modify the model to incorporate competition as

$$p_{n+1} - p_n = k_1 p_n - k_2 p_n (p_n - 1) \tag{6.25}$$

again ignoring effects due to migration. Here we are assuming $k_2 > 0$ and use the negative sign to reflect the fact that competition reduces the population. This equation can be simplified to

$$p_{n+1} - p_n = c_1 p_n - c_2 p_n^2 \tag{6.26}$$

This is the well-known *logistic* difference equation for population growth and it appears to correspond well to the growth of bacteria in agar jelly, for example.

Superficially we see that the difference between the model that does not model competition and the one that does is a quadratic term. A more fundamental difference is that Equation (6.22) is linear while Equation (6.26) is nonlinear. The only fixed point for Equation (6.22) is $\overline{p} = 0$. For Equation (6.26) there are now two fixed points $\overline{p_1} = 0$ and $\overline{p_2} = \frac{c_1}{c_2}$; see Figure 6.3. From the plot of $p_{n+1} - p_n$ it is clear that this new model predicts that the population will be limited, i.e., it can't grow unbounded to $\infty$ because as soon as $p_n > c_1/c_2$ then $\Delta p_n < 0$ so the population must decrease.

One is initially tempted to conclude that the equilibrium point $\overline{p_1} = 0$ is unstable while the equilibrium point $\overline{p_2} = c_1/c_2$ is stable. As we shall see in the numerical simulations this can be true, but for certain values of $c_1$ and $c_2$ the situation can be much more complicated including periodic and even chaotic solutions!

### 6.4.1 Systems of Equations and Competing Species

Now consider two species A and B whose populations are denoted $a_n$ and $b_n$, respectively. If we assume that these species have infinite resources and compete neither with themselves or each other then we would propose the simple *system* of difference equations

$$a_{n+1} - a_n = k_1 a_n$$

$$b_{n+1} - b_n = k_2 b_n$$

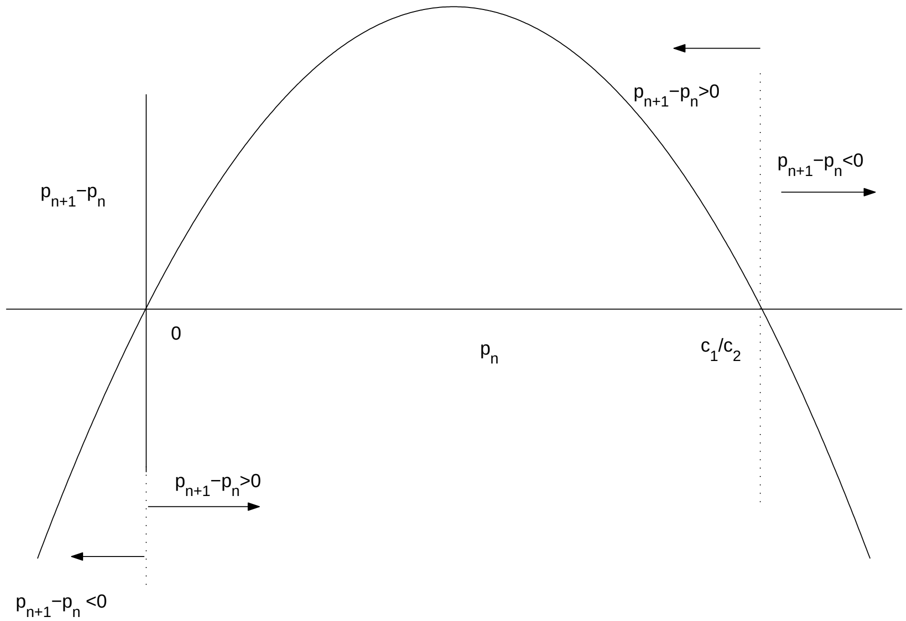


**FIGURE 6.3:** A plot of the change in population $p_{n+1} - p_n$ as a function of the population $p_n$.

This system is said to be *uncoupled* as the values of $a_n$ do not influence $b_n$ and, similarly, the values of $b_n$ do not influence $a_n$.

If species B eats the same kind of food species A does, but species A does not eat the same kind of food species B does we have the model

$$a_{n+1} - a_n = g_1 a_n - c_1 a_n b_n$$
$$b_{n+1} - b_n = g_2 b_n$$

If species A and B both like each others food we would employ the model

$$a_{n+1} - a_n = g_1 a_n - c_1 a_n b_n$$
$$b_{n+1} - b_n = g_2 b_n - c_2 a_n b_n$$

See Figure 6.4 for a numerical simulation of this system. Note that this nonlinear system does not have a closed form solution. For the parameters selected we see that even though species A initially has a lower population it appears to grow without bound while population B becomes extinct. Here we may conclude that species A is more fit than species B and consequently survives.

If species A and B compete both with each other and with themselves the population model would then become

$$a_{n+1} - a_n = g_1 a_n - c_1 a_n b_n - k_1 a_n^2$$
$$b_{n+1} - b_n = g_2 b_n - c_2 a_n b_n - k_2 b_n^2$$

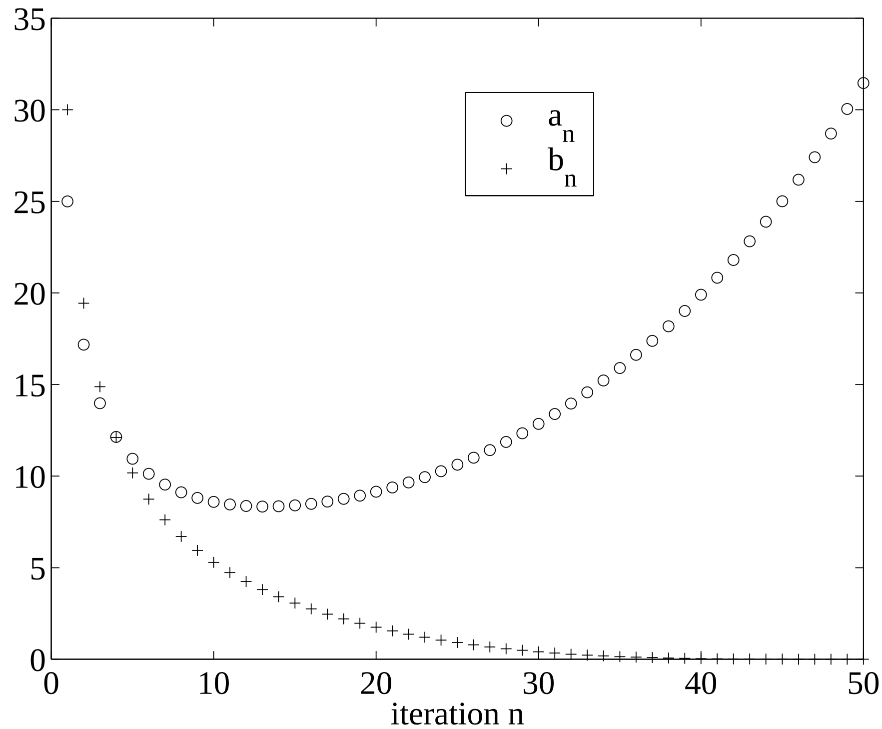


**FIGURE 6.4:** Competition for finite resources. We have selected the initial conditions $a_1 = 25$ and $b_1 = 30$. In addition the parameters were chosen to be $g_1 = .047, c_1 = .012, g_2 = .023, c_2 = .015$.

Notice that if population B becomes extinct while A survives then the model reduces to the logistic difference equation for a single species.

**Predator Prey Model.** Now consider modeling the interaction between natural predators and their prey. A classic example of this relationship is given by foxes and rabbits. The population of foxes and rabbits are intimately linked given that the rabbits are the food supply for the foxes. When the population of rabbits increases one can predict an associated, though possibly time lagged, increase in the number of foxes. Conversely, when the number of rabbits decreases the less food there is for the foxes. Of course an increase in the number of foxes will result in more rabbits being eaten and thus a reduction in the rabbit population.

Let's develop a model for this situation. First, denote the fox population by $f_n$ and the rabbit population by $r_n$. If we assume that in the absence of rabbits the fox population becomes extinct we have the model

$$\Delta f_n = -g_1 f_n$$

where the constant $g_1 > 0$. If rabbits are available, then they should contribute positively to a change in the fox population. It seems reasonable to assume that the increase in the fox population will be proportional to the number of fox and rabbit interactions which is given by the product $f_n r_n$. Thus, in the presence of rabbits we may model the change in the fox population to be

$$\Delta f_n = -g_1 f_n + c_1 f_n r_n$$

where the constant $c_1 > 0$

Now the rabbits should multiply in the absence of foxes

$$\Delta r_n = g_2 r_n$$

where the constant $g_2 > 0$. The impact of the foxes on the rabbits is presumably also proportional to the number of interactions but now this reduces the rabbit population.

$$\Delta r_n = g_2 r_n - c_2 f_n r_n$$

In summary we have the model

$$f_{n+1} = (1 - g_1) f_n + c_1 f_n r_n \tag{6.27}$$

$$r_{n+1} = (1 + g_2) r_n - c_2 f_n r_n \tag{6.28}$$

Note that we have omitted the competition amongst the foxes for the rabbits as well as the competition amongst the rabbits for their food. This is easily captured by extending the above system to

$$f_{n+1} = (1 - g_1) f_n + c_1 f_n r_n - d_1 f_n^2 \tag{6.29}$$

$$r_{n+1} = (1 + g_2) r_n - c_2 f_n r_n - d_2 r_n^2 \tag{6.30}$$

See Figure 6.5 for a simulation of the above equations. Note that the predicted oscillation is in fact there, however it is damped and the solution proceeds to a stable equilibrium.

## 6.5 EMPIRICAL MODELING

One may imagine that true observations, e.g., from populations in nature, will not be precise due to limitations in counting species in the wild. Thus, the data will contain what we refer to as a unknown *noise* component. In general, model selection can be arrived at by

1. Collect observations to build models
2. Propose models, e.g., predator prey or competing species
3. Compute model coefficients in each case
4. Compare models through validation and testing

Now we present the method of least squares as a means to determine our unknown model coefficients.

### 6.5.1 Non-Newtonian Fish?

Recall that Newton's Law of Cooling states that the temperature change in a body is proportional to the difference between the temperature of the body $T_n$ and the

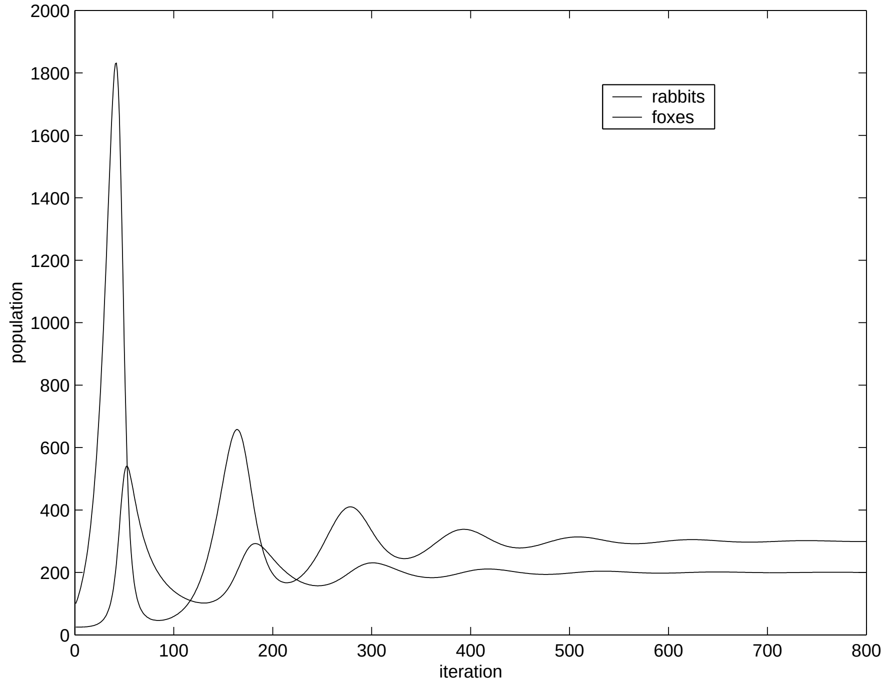


**FIGURE 6.5:** Simulation of preditor prey equations. We have selected the initial conditions $f_1 = 25$ and $r_1 = 100$. In addition the parameters were chosen to be $g_1 = 0.01, c_1 = 0.0001, g_2 = 0.1, c_2 = 0.0005,$ $d_1 = 0.0001$ and $d_2 = 0$.

surrounding temperature $M$, i.e., as a difference equation

$$\Delta T_n = k(M - T_n)$$

After repeatedly overcooking a certain kind of fish based on this law a frustrated cook has decided to take science into her own hands. She speculates that the actual law of cooking for this fish has the more general form

$$\Delta T_n = k(M - T_n)^\alpha$$

and that for certain types of foods, call them *Non-Newtonian foods*, that $\alpha \neq 1$.

To test her hypothesis, our cook measures the temperature of a fish every minute until it approaches the temperature of the oven which is set to 425 degrees F. The results of her data collection are shown in Figure 6.6.

Now $\Delta T_n$ is known since $T_n$ is known for $n = 1, \dots, 200$. Thus, for any $\alpha$ and $k$ we can compute a model error of

$$E(\alpha, k) = \sum_n \big(\Delta T_n - k(425 - T_n)^\alpha\big)^2$$

We recall from our previous work with least squares that computing $\alpha$ and $k$ requires differentiating the error term $E(\alpha, k)$ with respect to $\alpha$ and $k$. For this particular

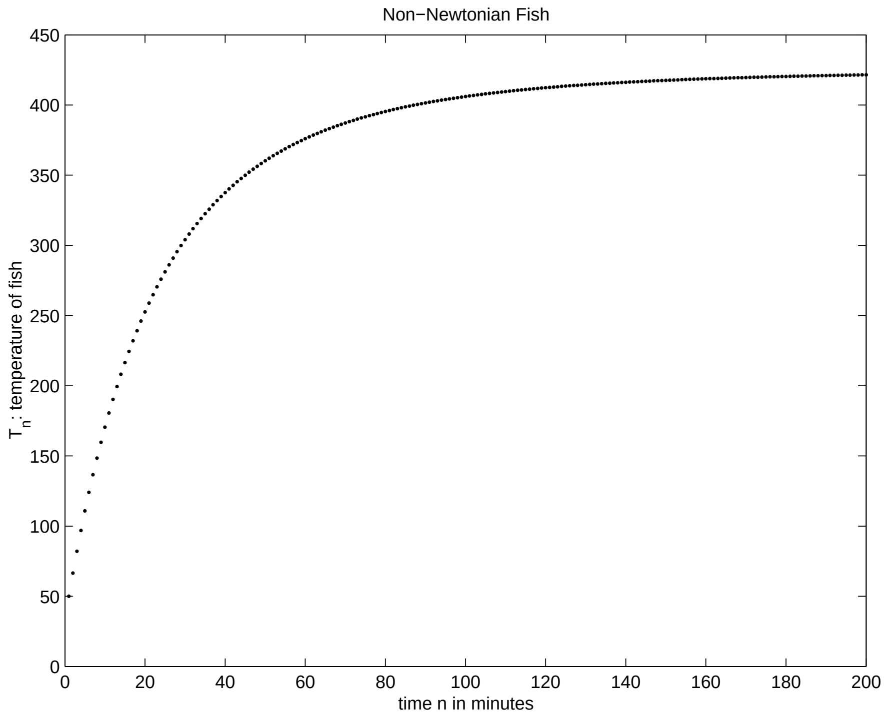


**FIGURE 6.6:** Observations of a Non-Newtonian fish. These are (synthetic) measurements of the temperature of the fish as a function of time.

model it is simpler to employ a logarithmic transformation

$$y_n = \ln \Delta T_n$$

$$b = \ln k$$

$$x_n = \ln(425 - T_n)$$

giving

$$E(\alpha, b) = \sum_n (y_n - b - \alpha x_n)^2$$

Differentiating these with respect to $\alpha$ and $b$ and setting the results equal to zero produces the equations

$$\begin{pmatrix} \sum_n x_n^2 & \sum_n x_n \\ \sum_n x_n & P \end{pmatrix} \begin{pmatrix} b \\ \alpha \end{pmatrix} = \begin{pmatrix} \sum_n y_n x_n \\ \sum_n y_n \end{pmatrix}$$

Solving these equations using *only the first 101 observations* $T_0, T_1, \ldots, T_{100}$ and the MATLAB code provided produces the results

$$\alpha = 1.25$$

and

$$k = 0.01$$

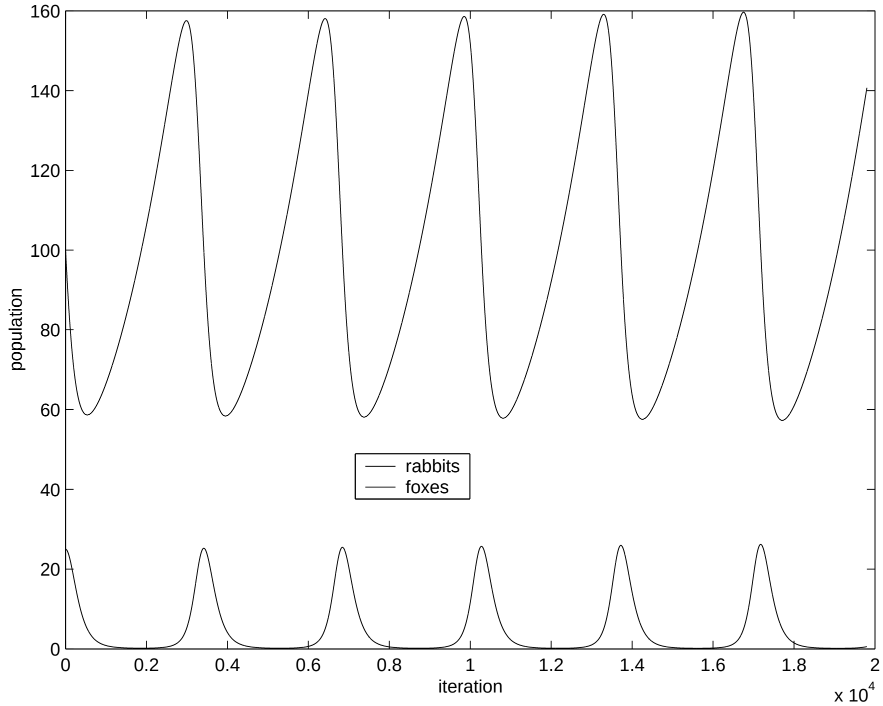


**FIGURE 6.7:** Simulation of preditor prey equations. We have selected the initial conditions $f_0 = 25$ and $r_0 = 100$. In addition the parameters were chosen to be $g_1 = .01, g_2 = .0005, c_1 = .0001, c_2 = .0001, d_1 = 0.0, d_2 = 0.0$

### 6.5.2 Predator or Prey?

Assume that the data in Figure 6.7 is provided. The goal is to see if we can calculate the coefficients of the preditor prey equations that will reproduce this data. Thus, given the tentative model

$$\Delta f_n = -g_1 f_n + c_1 f_n r_n$$
$$\Delta r_n = g_2 r_n - c_2 f_n r_n$$

the points $\{f_n, r_n\}$ are now observations while the equation coefficients $\{g_1, g_2, c_1, c_2\}$ are to be determined.

The least squares error is now

$$E(g_1, c_1, g_2, c_2) = \sum_n (\Delta f_n + g_1 f_n - c_1 f_n r_n)^2 + \sum_n (\Delta r_n - g_2 f_n + c_2 f_n r_n)^2 \quad (6.31)$$

Setting

$$\frac{\partial E}{\partial g_1} = \frac{\partial E}{\partial c_1} = \frac{\partial E}{\partial g_2} = \frac{\partial E}{\partial c_2} = 0$$

produces the necessary conditions for a minimum error. Taking the uncoupled equations for $c_1$ and $g_1$ we have

$$\begin{pmatrix} -\sum_n f_n^2 & \sum_n f_n^2 r_n \\ -\sum_n f_n^2 r_n & \sum_n f_n^2 r_n^2 \end{pmatrix} \begin{pmatrix} g_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sum_n (\Delta f_n) f_n \\ \sum_n (\Delta f_n) f_n r_n \end{pmatrix} \tag{6.32}$$

These must be solved simultaneously with the uncoupled conditions for $c_2$ and $g_2$, i.e.,

$$\begin{pmatrix} -\sum_n r_n^2 & -\sum_n r_n^2 f_n \\ \sum_n r_n^2 f_n & -\sum_n f_n^2 r_n^2 \end{pmatrix} \begin{pmatrix} g_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} \sum_n (\Delta r_n) r_n \\ \sum_n (\Delta r_n) f_n r_n \end{pmatrix} \tag{6.33}$$

Solving these equations produces the exact coefficients that were used to generate the data. In principal, this procedure may be applied to direct observations from nature. One may conclude if a model *fits* the data and, if so, which species plays which role, i.e., by examining the computed signs of $c_1, c_2, g_1$ and $g_2$ one may infer which species is the predator and which is the prey. See the MATLAB code for these equations in the Appendix.

## PROBLEMS

**6.1.** Consider the following equations and identify as

- linear or nonlinear
- homogeneous or nonhomogenous
- which order

**(a)** $x_{n+1}^2 + x_n = 1.$
**(b)** $x_{n+1} = x_{n-1} + 2$
**(c)** $x_{n+1} = \sin(x_{n-1})$
**(d)** $x_{n+3} = x_{n+1} + x_{n-3} + n^2$

**6.2.** Determine particular solutions to the following equations
**(a)** $x_{n+1} = x_n + 1$
**(b)** $x_{n+1} = 5x_n + n^2$
**(c)** $x_{n+1} = \frac{x_n}{2} + 6^n$

**6.3.** Show that the real and imaginary parts of a complex solution to a linear difference equation are also solutions to the same difference equation.

**6.4.** Determine general solutions to the following equations
**(a)** $x_{n+1} = x_n + 1$
**(b)** $x_{n+1} = 5x_n + n^2$
**(c)** $x_{n+1} = \frac{x_n}{2} + 6^n$
**(d)** $x_{n+1} = \frac{x_n}{2} + 4n^2 + 2n + 1$

**6.5.** You currently have \$5000 in a savings account that pays 6% interest per year. Interest is compounded monthly. You add another \$200 each month. What do you have on your savings account after five years, and what is the total interest earned during these five years?

**6.6.** You owe \$500 on a credit card that charges 1.5% interest in each month. You can pay \$50 each month and make no new charges. When is your loan paid off and what is your last payment? How much interest have you paid?

**6.7.** Your parents are considering a 30–year \$100, 000 mortgage at an annual interest rate of 6%. What is the monthly payment, and what will be the total interest paid?

**6.8.** Mary receives \$5000 as graduation present from her parents when graduating from High School. She deposits the money on a savings account at an annual interest rate of 3%. Interest is compounded monthly. Before going to college she works for three years, and during this time she deposits each month a certain amount on the savings account. She plans to withdraw \$1200 each month in her first year on college, and to increase the monthly withdrawal in each of the following three years by \$100 (in her fourth year on college she withdraws \$1500 each month). What must the monthly payment during the first three years be in order that after Mary's four years on college the balance on the savings account is zero again, and what is the total interest Mary has earned after the seven years?

**6.9.** Redo Example 6.11 for the case that the annual salary $s_m$ is paid during the first nine months (monthly payment $s_m/9$) in each year, i.e., there is no income and hence no payment on the retirement savings account during the last three months of the year. (This is the situation of university professors if they don't have additional income from grants.)

**6.10.** Assume the temperature of a roast in the oven increase at a rate proportional to the difference between the oven (set to 400 degrees F) and the roast. If the roast enters the oven at 50 degrees F and is measured one hour later to be at 90 when should the table be set if the eating temperature is 166 degrees F? Hint: write down the difference equation and solve analytically.

**6.11.** *Computer.* This question concerns numerically exploring the solutions of the equation

$$p_{n+1} = p_n + \alpha p_n(1 - p_n)$$

First determine all the *equilibrium* solutions of this difference equation by setting $\overline{p} = p_{n+1} = p_n$. Now investigate the stability of these equilibrium numerically. Consider the initial conditions

- $p_0 = 0$
- $p_0 = 0.0001$
- $p_0 = 2$

Numerically simulate the difference equation using the following values of $\alpha$

- $\alpha = .1$
- $\alpha = .7$
- $\alpha = 1.2$

Describe your results and comment on the stability of the equilibrium you found. Provide plots of all your results. It will make your comparisons easier if you plot all the results for one value of $\alpha$ on a single graph.

**6.12.** *Computer.* This question concerns numerically exploring the solutions of the equation

$$p_{n+1} = p_n + 0.1p_n(1 - p_n)(2 - p_n)$$

First determine all the *equilibrium* solutions of this difference equation. Numerically simulate the difference equation using the following initial conditions

- $p_0 = 0$
- $p_0 = 0.0001$
- $p_0 = .9999$
- $p_0 = 1$
- $p_0 = 1.0001$
- $p_0 = 1.9999$
- $p_0 = 2$
- $p_0 = 2.0001$

Describe your results and comment on the stability of the equilibrium you found. Provide plots of all your results. It will make your comparisons easier if you plot all the results on a single graph.

**6.13.** *Computer.* Simulate the fourth order difference equation

$$p_{n+4} = \sin(p_{n+3} + p_{n+2} + p_{n+1} - p_n) + 2$$

and compare to the related equation

$$p_{n+4} = \sin(p_{n+2} + p_{n+1} - p_n) + 2$$

using the initial coditions $p_1 = 6, p_2 = 1, p_3 = 2.5, p_4 = -3$. Explore modifications to these difference equations and see if you can find any interesting behavior. For example, what is the effect of varying the nonhomogeneous term? Plot your results in each case for 100 iterations.

**6.14.** *Computer.* Consider the system of difference equations

$$x_{n+1} = 0.3x_n + 0.8y_n$$

$$y_{n+1} = 0.7x_n + 0.2y_n$$

Simulate these equations numerically for a variety of initial conditions and attempt to determine any stable equilibrium. Verify that you have actually determined an equilibrium solution by substituting into the original system. (Note that the equilibrium solution in this problem actually depends on the initial condition.)

**(a)** How do the solutions change if you modify the first coefficient from 0.30 to 0.31?

**(b)** How do the solutions change if you modify the first coefficient from 0.30 to 0.31 *and* modify the 0.7 coefficient to 0.69.

**(c)** Compare the results in part a) and b). Can you explain?

**6.15.** Consider the difference equation

$$x_{n+2} + \alpha x_{n+1} + \beta x_n = 0$$

where it is assumed that $\alpha^2 - 4\beta = 0$. Show that $x_n = (-\frac{\alpha}{2})^n n$ is a solution.

**6.16.** Find the linear second order nonhomogeneous difference equation relating the price $p_{2n+2}$ and $p_{2n}$ in the cobweb model. Solve this equation and produce a convergence criterion. What does the equilibrium price converge to? Check your result by computing the point of intersection of the supply and demand curves.

**6.17.** Determine analytical solutions to the following difference equations assuming in each case that $x_1 = 1$ and $x_0 = -1$. Plot your results.

a) $x_{n+2} + 3x_{n+1} + x_n = 0$

b) $10x_{n+2} + x_{n+1} + x_n = 0$

c) $x_{n+2} + \sqrt{3}x_{n+1} + \frac{3}{4}x_n = 0$

**6.18.** Extend the population model with pairwise competition to include competition among groups of three. Furthermore, assume that the competition among groups of three is more intense than competition between pairs. Identity the new equilibrium solution(s). Use a plot of $\Delta p_n$ versus $p_n$ to argue whether the model predicts a bounded population.

**6.19.** Consider a clam population that obeys the logistic difference Equation (6.26). Modify this equation to account for constant harvesting of the clams. By computing the new equilibrium points of the population model describe the impact of harvesting on the clam population.

**6.20.** Consider three species A, B, C and the evolution of their populations $a_n, b_n$ and $c_n$.

- Species A eats B and C
- Species B eats neither A nor B
- Species C eats only A.
- Species B eats waste products produced by species A and B.
- The population of both species A and B increase in the absence of other species.
- The population of species C decreases in the absence of A and B.

- Species C is competes with itself for food while this is not true for species A and B.

Write down a system of three coupled difference equations modeling the populations of the three species.

**6.21.** *Computer.* Provide a model for the bee colony population data in Table 6.2. What does your model predict the long-term population to be?

| day | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| number | 20 | 25 | 60 | 85 | 111 | 146 | 177 | 182 | 184 | 171 |
| day | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| number | 179 | 167 | 161 | 146 | 159 | 154 | 162 | 166 | 166 | 168 |

**TABLE 6.2:** Bee colony population data.

**6.22.** Find the equilibria of the difference equation

$$p_{n+1} = p_n - 0.1p_n(1 - p_n)(2 - p_n)$$

and determine which of them are stable.

**6.23.** *Computer.* Numerically compute and plot 50 iterates of the difference equation

$$p_{n+1} = p_n - 0.1p_n(1 - p_n)(2 - p_n)$$

for each of the initial conditions
**(a)** (a) $p_0 = 0.9$
**(b)** $p_0 = 1.1$.
Is the behavior of the iterates consistent with the stability calculation of Problem 6.22?

**6.24.** *Computer.* Find all the equilibrium solutions of the logistic difference equation

$$x_{n+1} = rx_n(1 - x_n)$$

as a function of $r$. Letting $x_0 = 0.2$ numerically iterate this difference equation for 200 iterations for the following values of $r$:

- $r = 2$
- $r = 3.2$
- $r = 3.8282$
- $r = 3.83$

Plot your results $x_n$ as a function of $n$ for each case and comment. Does this seem like a reasonable model for a population?

**6.25.** Consider the logistic difference equation with $r > 0$:

$$p_{n+1} = rp_n(1 - p_n).$$

**(a)** Show that $\overline{p}_1 = 0$ is an equilibrium.
**(b)** Find the second equilibrium $\overline{p}_2(r)$. For which values of $r$ is $\overline{p}_2(r) \geq 0$?
**(c)** For which values of $r$ is $\overline{p}_1 = 0$ stable?
**(d)** For which values of $r$ is $\overline{p}_2(r)$ stable?

**6.26.** *Computer.* Numerically compute and plot 50 iterates of the difference equation

$$p_{n+1} = rp_n(1 - p_n)$$

for $p_0 = 0.5$ and each of the following values of $r$:
**(a)** $r = 0.8$
**(b)** $r = 2.9$
**(c)** $r = 3.1$
**(d)** $r = 3.5$
**(e)** $r = 3.9$.
Describe the behavior of the iterates and relate it, where possible, to the stability calculation of Problem 6.25.

**6.27.** *Computer.* Use a least squares approach to determine $k$ in Newton's Law of cooling

$$T_{n+1} = T_n + k(M - T_n)$$

using the data generated by our empirical fish model

$$T_{n+1} = T_n + 0.01(M - T_n)^{1.25}$$

First generate 200 points using this equation and compute $k$ based on these points. Now predict the next 200 points and calculate the error. If a fish is well cooked at 170 degrees F how long does each model predict it will take to cook the fish? Use the values $M = 425$ and $T_0 = 50$.

**6.28.** *Computer.* Using the data provided in Table 6.3 estimate via least squares the coeffients $c_1, c_2, d_1, d_2$ in the model

$$a_{n+1} = a_n + c_1 a_n + d_1 b_n$$

$$b_{n+1} = b_n + c_2 a_n + d_2 b_n$$

Include your equations for the unknown coefficients in your write-up.

| $n$ | $a_n$ | $b_n$ |
|---|---|---|
| 1 | 15.00 | 45.00 |
| 2 | 30.00 | 30.00 |
| 3 | 24.00 | 36.00 |
| 4 | 26.40 | 33.60 |
| 5 | 25.44 | 34.56 |
| 6 | 25.82 | 34.18 |
| 7 | 25.67 | 34.33 |
| 8 | 25.73 | 34.27 |
| 9 | 25.71 | 34.29 |
| 10 | 25.72 | 34.28 |
| 11 | 25.71 | 34.29 |

**TABLE 6.3:** Did this data come from a linear system?

**6.29.** Extend the Equations (6.32) and (6.33) provided for computing the coefficients $c_1, c_2, g_1, g_2$ for the predator-prey model with no intra-species competition given by Equation (6.27) to the case of Equations 6.29 where intraspecies competition is accounted for. Your equations should now provide estimates for $c_1, c_2, g_1, g_2, d_1, d_2$

**6.30.** Consider the differential equation for the unforced damped nonlinear pendulum

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \sin x = 0$$

where $x(t)$ represents the angular displacement from the equilibrium in radians. Using the expressions for the numerical estimates of the derivatives

$$\frac{d^2x}{dt^2} = \frac{x_{n+1} + x_{n-1} - 2x_n}{(\Delta t)^2}$$

and

$$\frac{dx}{dt} = \frac{x_n - x_{n-1}}{\Delta t}$$

where $x_n \equiv x(n\Delta t)$.

**(a)** Show that the differential equation can be approximated by the second order difference equation

$$x_{n+1} = (2 - \alpha\Delta t)x_n + (\alpha\Delta t - 1)x_{n-1} - (\Delta t)^2 \sin(x_n) \qquad (6.34)$$

**(b)** Simulate this difference equation for 1000 iterations using the values $\Delta t = 0.05, \alpha = 0.1, x_1 = 0, x_2 = 0.0001$ and plot your result. Repeat this calculation for $x_1 = 2, x_2 = 2.0001$ and compare your results.

**(c)** Redo this simulation using the *small angle approximation* $\sin x = x$, i.e., simulate

$$x_{n+1} = (2 - \alpha\Delta t)x_n + (\alpha\Delta t - 1)x_{n-1} - (\Delta t)^2 x_n \qquad (6.35)$$

using the values $\Delta t = 0.05, \alpha = 0.1, x_1 = 0, x_2 = 0.0001$ and plot your result. Again, repeat this calculation for $x_1 = 2, x_2 = 2.0001$ and compare your results with those found in part (c).

**(d)** Rewrite the second order Equation (6.34) as a system of two first order equations via the substitution $y_{n+1} = x_n$ and determine all equilibria. Note that the equilibria can also be determined directly from Equation (6.34).

**(e)** By computing the eigenvalues of the Jacobian matrix of this system, ascertain which equilibria are stable and unstable. Discuss.

**6.31.** Repeat parts (d) and (e) of Problem 6.30 for the small angle approximation Equation (6.35) and compare.

**6.32.** Analytically solve the linear difference equation from the previous problem

$$x_{n+1} = (2 - \alpha\Delta t)x_n + (\alpha\Delta t - 1)x_{n-1} - (\Delta t)^2 x_n$$

and compare with your numerical simulation above. For simplicity you may take $\Delta t = 0.05, \alpha = 0.1, x_1 = 2, x_2 = 2.0001$.

**6.33.** Analytically solve the linear nonhomogeneous difference equation

$$x_{n+1} = (2 - \alpha\Delta t)x_n + (\alpha\Delta t - 1)x_{n-1} - (\Delta t)^2 x_n + 0.01\sin(n/50)$$

Simulate this problem numerically and compare with your analytical solution for 2000 iterations. Can you identify a transient (i.e., a term that goes to zero) and steady state (persistent) components of your solution? Again, for simplicity you may take $\Delta t = 0.05, \alpha = 0.1, x_1 = 2, x_2 = 2.0001$. Hint: combine your solution to the homogeneous problem found above with a particular solution of the form

$$p_n = A\cos(n/50) + B\sin(n/50)$$

Solve for the undetermined coefficients $A$ and $B$.

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