C H A P T E R 2

Qualitative Modeling with Functions

It is often surprising that very simple mathematical modeling ideas can produce results with added value. Indeed, the solutions may be elegant and provide quality of understanding that obviates further exploration by more technical or complex means. In this chapter we explore a few simple approaches to qualitatively modeling phenomena with well-behaved functions.

2.1 MODELING SPECIES PROPAGATION

This problem concerns the factors that influence the number of species existing on an island. The discussion is adapted from [1].

One might speculate that factors affecting the number of species could include

- Distance of the island from the mainland
- Size of the island

Of course limiting ourselves to these influences has the dual effect of making a tractable model that needs to be recognized as omitting many possible factors.

The number of species may increase due to new species discovering the island as a suitable habitat. We will refer to this as the *migration rate*. Alternatively, species may become extinct due to competition. We will refer to this as the *extinction rate*. This discussion will be simplified by employing an aggregate total for the number of species and not attempting to distinguish the nature of each species, i.e., birds versus plants.

Now we propose some basic modeling assumptions that appear reasonable.

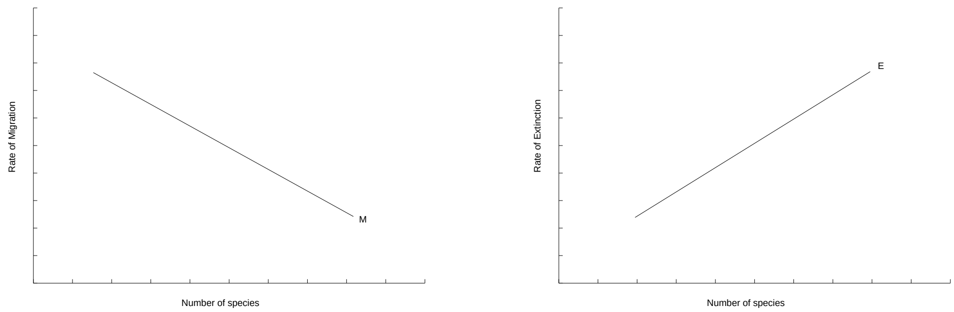
> *The migration rate of new species decreases as the number of species on the island increases.*

The argument for this is straight forward. The more species on an island the smaller the number of new species there is to migrate. See Figure 2.1 (a) for a qualitative picture.

> *The extinction rate of species increases as the number of species on the island increases.*

Clearly the more species there are the more possibilities there are for species to die out. See Figure 2.1 (b) for a qualitative picture.

If we plot the extinction rate and the migration rate on a single plot we identify the point of intersection as an equilibrium, i.e., the migration is exactly offset by the extinction and the number of species on the island is a constant. We



(a) Migration curve. (b) Extinction curve

FIGURE 2.1: Qualitative form of the migration and extinction curves.

will assume in this discussion that we are considering islands for which the number of species is roughly constant over time, i.e., they are in a state of equilibrium.

Now we consider whether this simple model provides any added value. In particular, can it be used to address our questions posed at the outset.

First, what is the effect of the distance of the island from the mainland on the number of bird species? One can characterize this effect by a shift in the migration curve. The further the island is away from the mainland, the less likely a species is to successfully migrate. Thus the migration curve is shifted down for *far* islands and shifted up for *near* islands. Presumably, this distance of the island from the mainland has no impact on the extinction curve. Thus, by examining the shift in the equilibrium, we may conclude that the number of species on an island decreases as the island's distance from the mainland increases. See Figure 2.2.

Note in this model we assume that the time-scales are small enough that new species are not developed via evolution. While this may seem reasonable there is evidence that in some extreme climates, such as those found in the Galapagos Islands, variation may occur over shorter periods. There have been 140 different species of birds

2.2 SUPPLY AND DEMAND

In this section we sketch a well-known concept in economics, i.e., supply and demand. We shall see that relatively simple laws, when taken together, afford interesting insight into the relationship between producers and consumers. Furthermore, we may use this framework to make predictions such as

- What is the impact of a tax on the sale price?
- What is the impact an increase in employees wages on sales price? Can the owner of the business pass this increase on to the consumer?

Law of Supply: An increase in the price of a commodity will result in an increase of the amount supplied.

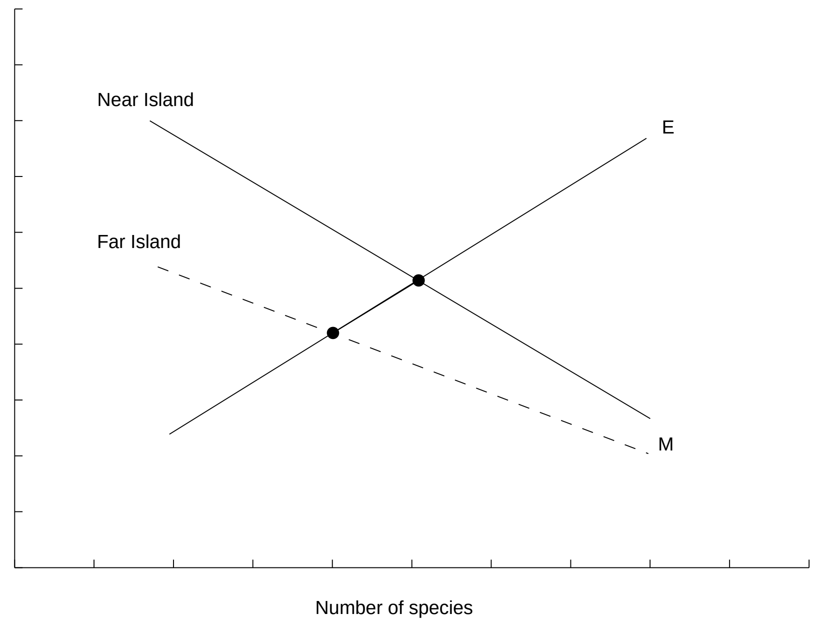


FIGURE 2.2: The effect of distance of the island from the mainland is to shift the migration curve. Consequently the equilibrium solution dictates a smaller number of species will be supported for islands that are farther away from the mainland.

> *Law of Demand:* If the price of a commodity increases, then the quantity demanded will decrease.

Thus, we may model the supply curve qualitatively by a monotonically increasing function. For simplicity we may assume a straight line with positive slope. Analogously, we may model the demand curve qualitatively by a monotonically decreasing function, which again we will take as a straight line.

A flat demand curve may be interpreted as consumers being very sensitive to the price of a commodity. If the price goes up just a little, then the quantity in demand goes down significantly. Steep and flat supply and demand curves all have similar qualitative interpretations (see the problems).

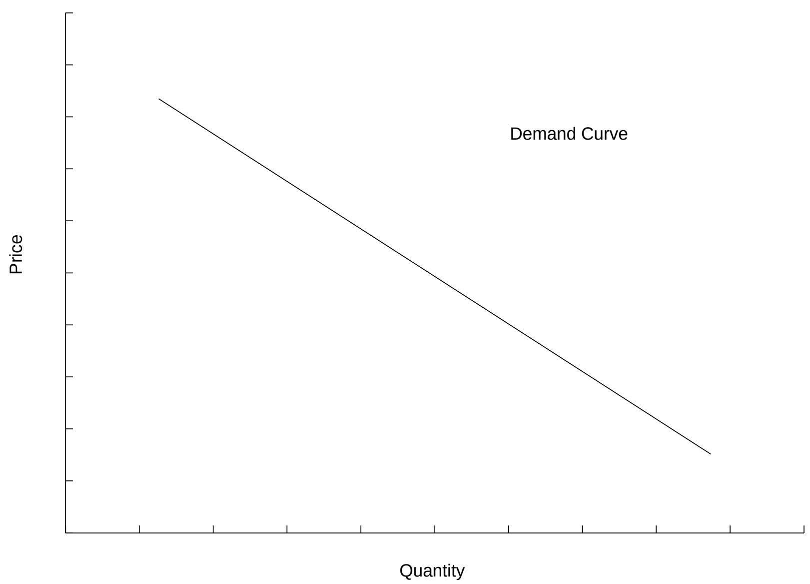
2.2.1 Market Equilibrium

Given a supply curve and a demand curve we may plot them on the same axis and note their point of intersection (q_*, p_*). This point is special for the following reason:

- The seller is willing to supply q_* at the price p_*
- The demand is at the price p_* is q_*

So both the supplier(s) and the purchaser(s) are happy economically speaking.

(a) Supply curve



(b) Demand curve

FIGURE 2.3: (a) Qualitative form of supply and demand curves.

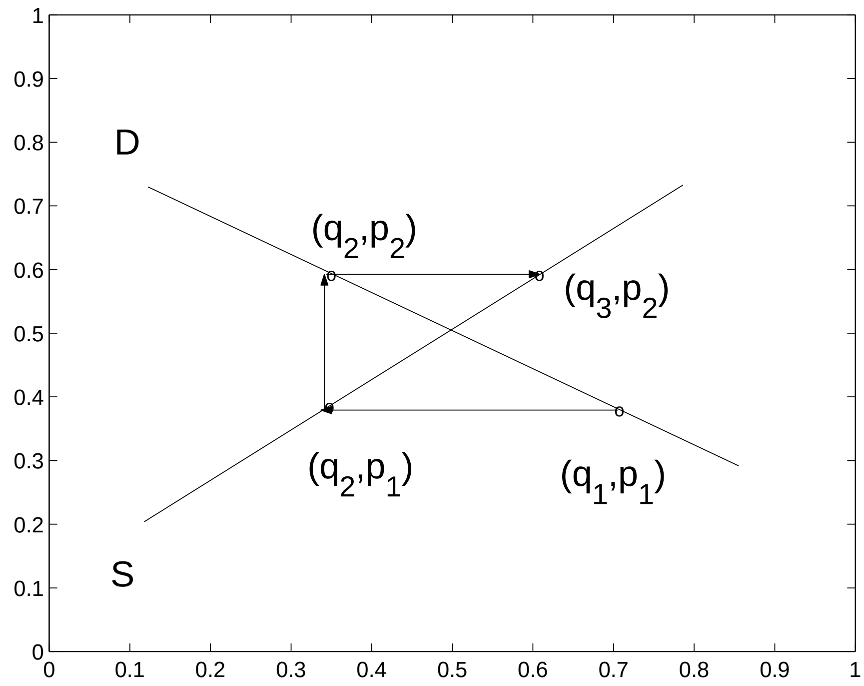


FIGURE 2.4: The cobweb model illustrating a sequence of market adjustments.

2.2.2 Market Adjustment

Of course, in general markets do not exist in the perfect economic utopia described above. We may model the market adjustment as a sequence of points on the demand and supply curves.

Based on market research it is estimated that consumers will demand a quantity q_1 at a price p_1. The supply and demand curves will permit a prediction of how the market will evolve. For simplicity, we will assume that the initial point (q_1, p_1) is on the demand curve to the right of the equilibrium point.

At the price p_1 the supplier looks to his supply curve and proposes to sell a reduced quantity q_2. Thus we move from right to left horizontally. Note that moving vertically to the supply curve would not make sense as this would correspond to offering the quantity q_1 at an increased price. These goods will not sell at this price.

From the point (q_1, p_2) the consumer will respond to the new reduced quantity q_2 by being willing to pay more. This corresponds to moving vertically upward to the new point (q_2, p_2) on the supply curve.

Now the supplier adjusts to the higher price being paid in the market place by increasing the quantity produced to q_3. This process then continues, in theory, until an equilibrium is reached. It is possible that this will never happen, at least not without a basic adjustment to the shape of either the supply of demand curves, for example through cost cutting methods such as improved efficiency, or layoffs.

2.2.3 Taxation

The effect of a new tax on a product is to shift the demand curve down because consumers will not be willing to pay as much for the product (before the tax). Note that this leads to a new equilibrium point which reduces the price paid to the seller per item and reduces the quantity supplied by the producer. Thus one may conclude from this picture that the effect of a tax on alcohol is to reduce consumption as well as profit for the supplier. See Figure 2.5.

2.3 MODELING WITH PROPORTION AND SCALE

In the previous sections we have considered how simple functions may be employed to qualitatively model various situations and produce added value. Now we turn to considerations that assist in determining the nature of these functional dependencies in more complex terms.

2.3.1 Proportion

If a quantity y is *proportional* to a quantity x then we write

$$y \propto x$$

by which is meant

$$y = kx$$

for some constant of proportionality k.

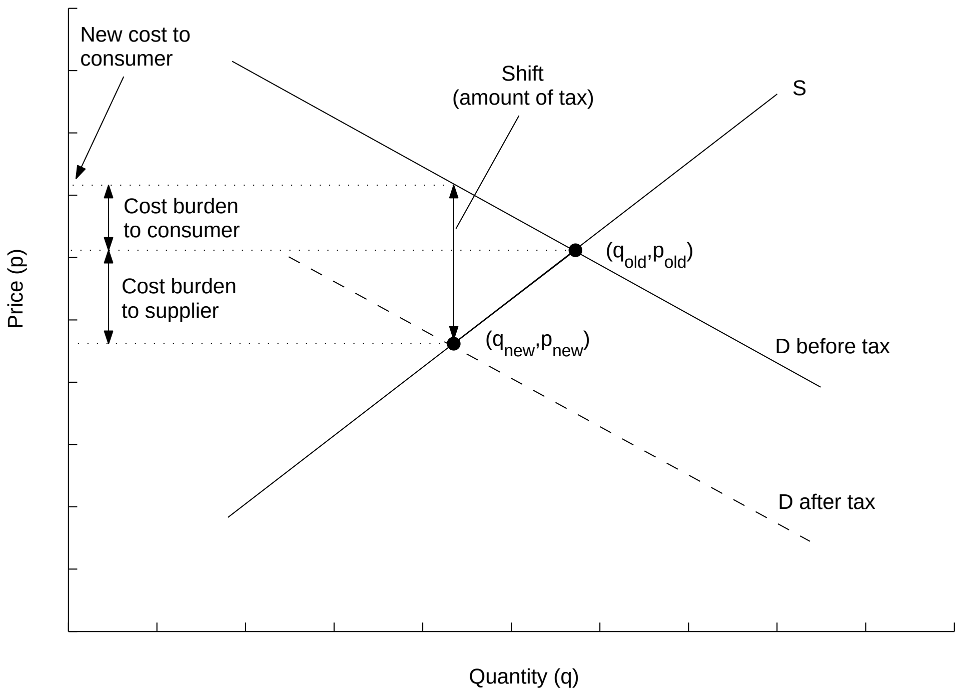


FIGURE 2.5: A tax corresponds to a downwards shift in the demand curve.

EXAMPLE 2.1

In 1678 Robert Hooke proposed that the restoring force F of a spring is proportional to its elongation e, i.e.,

$$F \propto e$$

or,

$$F = ke$$

where k is the *stiffness* of the spring.

Note that the property of proportionality is symmetric, i.e.,

$$y \propto x \rightarrow x \propto y \tag{2.1}$$

and transitive, i.e.,

$$y \propto x \quad \text{and} \quad z \propto y \rightarrow z \propto x \tag{2.2}$$

EXAMPLE 2.2

If $y = kx + b$ where k, b are constants, then

$$y \not\propto x$$

but

$$y - b \propto x$$

Inverse proportion. If $y \propto 1/x$ then y is said to be *inversely* proportional to x.

EXAMPLE 2.3

If y varies inversely as the square-root of x then

$$y = \frac{k}{\sqrt{x}}$$

Joint Variation. The volume of a cylinder is given by

$$V = \pi r^2 h$$

where r is the radius and h is the height. The volume is said to vary *jointly* with r^2 and h, i.e.,

$$V \propto r^2 \quad \text{and} \quad V \propto h$$

EXAMPLE 2.4

The volume of a given mass of gas is proportional to the temperature and inversely proportional to the pressure, i.e., $V \propto T$ and $V \propto 1/P$, or,

$$V = k\frac{T}{P}$$

EXAMPLE 2.5

Frictional drag due to the atmosphere is jointly proportional to the surface area S and the velocity v of the object.

Superposition of Proportions. Often a quantity will vary as the sum of proportions.

EXAMPLE 2.6

The stopping distance of a car when an emergency situation is encountered is the sum of the reaction time of the driver and the amount of time it takes for the breaks to dissipate the energy of the vehicle. The reaction distance is proportional to the velocity. The distance travelled once the breaks have been hit is proportional to the velocity squared. Thus,

$$\text{stopping distance} = k_1 v + k_2 v^2$$

EXAMPLE 2.7

Numerical error in the computer estimation of the center difference formula for the derivative is given by

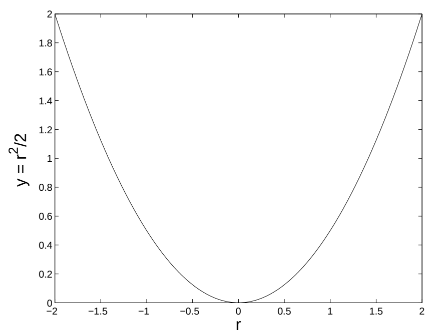
$$e(h) = \frac{c_1}{h} + c_2 h^2$$

where the first term is due to roundoff error (finite precision) and the second term is due to truncation error. The value h is the distance δx in the definition of the derivative.

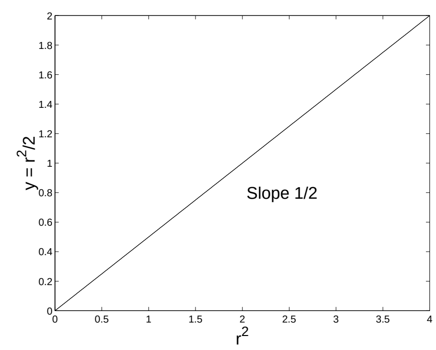
Direct Proportion. If

$$y \propto x$$

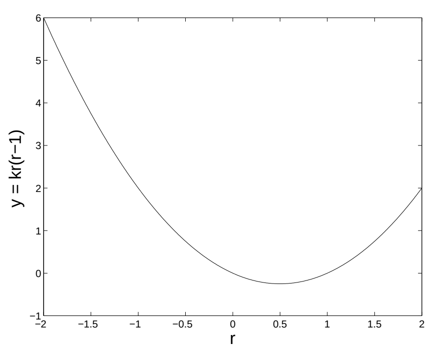
we say y varies in *direct* proportion to x. This is not true, for example, if $y \propto r^2$. On the other hand, we may construct a direct proportion via the obvious change of variable $x = r^2$. This simple trick always permits the investigation of the relationship between two variables such as this to be recast as a direct proportion.



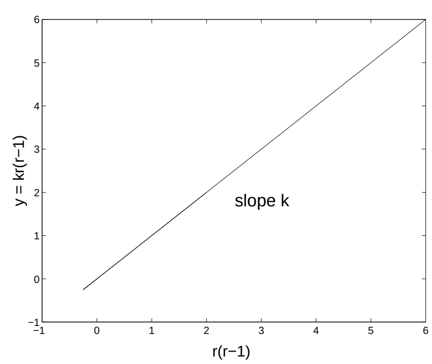
(a) Plot of y against r for $y = r^2/2$.



(b) Plot of $y = r^2/2$ against r^2.



(c) Plot of y against r for $y = kr(r+1)$.



(d) Plot of $y = kr(r+1)$ against$r(r+1)$

FIGURE 2.6: Simple examples of how a proportion may be converted to a direct proportion.

2.3.2 Scale

Now we explore how the size of an object can be represented by an appropriate length scale if we restrict our attention to replicas that are *geometrically similar*. For example, a rectangle with sides l_1 and w_1 is geometrically similar to a rectangle with sides l_2 and w_2 if

$$\frac{l_1}{l_2} = \frac{w_1}{w_2} = k \tag{2.3}$$

As the ratio $\kappa = l_1/w_1$ characterizes the geometry of the rectangle it is referred to as the *shape factor*. If two objects are geometrically similar, then it can be shown that they have the same shape factor. This follows directly from multiplying Equation (2.3) by the factor l_2/w_1, i.e.,

$$\frac{l_1}{w_1} = \frac{l_2}{w_2} = k\frac{l_2}{w_1}$$

Characteristic Length.

Characteristic length is useful concept for characterizing a family of geometrically similar objects. We demonstrate this with an example.

Consider the area of a rectangle of side l and width w where l and w may vary under the restriction that the resulting rectangle be geometrically similar to the rectangle with length l_1 and width w_1. An expression for the area of the varying triangle can be simplified as a consequence of the constraint imposed by geometric similarity. To see this

$$\begin{aligned} A &= lw \\ &= l(\frac{w_1 l}{l_1}) \\ &= \kappa l^2 \end{aligned}$$

where $\kappa = w_1/l_1$, i.e., the shape factor. See Figure 2.7 for examples of characteristic lengths for the rectangle.

EXAMPLE 2.8

Watering a farmer's rectangular field requires an amount of area proportional to the area of the field. If the characteristic length of the field is doubled, how much additional water q will be needed, assuming the new field is geometrically similar to the old field? Solution: $q \propto l^2$, i.e., $q = kl^2$. Hence

$$q_1 = kl_1^2$$

$$q_2 = kl_2^2$$

Taking the ratio produces

$$\frac{q_1}{q_2} = \frac{l_1^2}{l_2^2}$$

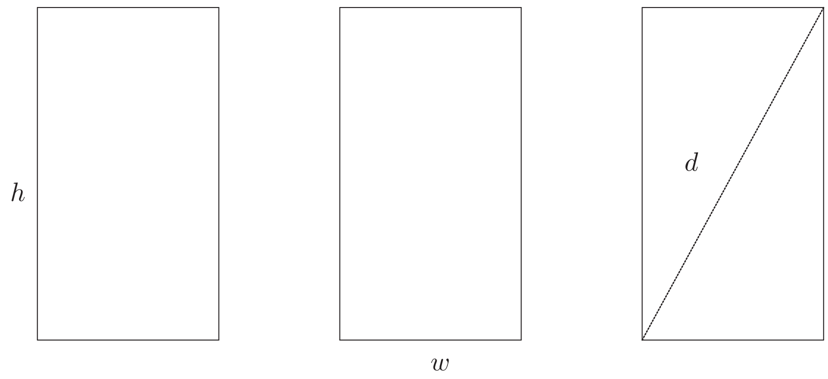


FIGURE 2.7: The height l_1, the width l_2 and the diagonal l_3 are all characteristic lengths for the rectangle.

Now if $q_2 = 100$ acre feet of water are sufficient for a field of length $l_2 = 100$, how much water will be required for a field of length $l_1 = 200$? Sol.

$$q_1 = q_2 \frac{l_1^2}{l_2^2} = 100 \frac{200^2}{100^2} = 400 \,\text{acre feet} \quad \square$$

EXAMPLE 2.9

Why are gymnasts typically short? It seems plausible that the ability A, or natural talent, of gymnast would be proportional to strength and inversely proportional to weight, i.e.,

$$A \propto \text{strength}$$

and

$$A \propto \frac{1}{\text{weight}}$$

and taken jointly

$$A \propto \frac{\text{strength}}{\text{weight}}$$

One model for strength is that the strength of a limb is proportional to the cross-sectional area of the muscle. The weight is proportional to the volume (assuming constant density of the gynmast). Now, assuming all gymnasts are geometrically similar with characteristic length l

$$\text{strength} \propto \text{muscle area} \propto l^2$$

and

$$\text{weight} \propto \text{volume} \propto l^3$$

so the ability A follows

$$A \propto \frac{l^2}{l^3} \propto \frac{1}{l}$$

So shortness equates to a talent for gymnastics. This problem was originally introduced in [2]. □

EXAMPLE 2.10

Proportions and terminal velocity. Consider a uniform density spherical object falling under the influence of gravity. The object will travel will constant (terminal) velocity if the accelerating force due to gravity $F_g = mg$ is balance exactly by the decelerating force due to atmospheric friction $F_d = kSv^2$; S is the cross-sectional surface area and v is the velocity of the falling object. Our equilibrium condition is then

$$F_g = F_d$$

Since surface area satisfies $S \propto l^2$ it follows $l \propto S^{1/2}$. Given uniform density $m \propto w \propto l^3$ so it follows $l \propto m^{1/3}$. Combining proportionalities

$$m^{1/3} \propto S^{1/2}$$

from which it follows by substitution into the force equation that

$$m \propto m^{2/3}v^2$$

or, after simplifying,

$$v \propto m^{1/6}$$

□

EXAMPLE 2.11

In this example we will attempt to model observed data displayed in Table 2.1 that relates the heart rate of mammals to there body weight. From the table we see that we would like to relate the heart rate as a function of body weight. Smaller animals have a faster heart rate than larger ones. But how do we estimate this proportionality?

We begin by assuming that all the energy E produced by the body is used to maintain heat loss to the environment. This heat loss is in turn proportional to the surface area s of the body. Thus,

$$E \propto s$$

The energy available to the body is produced by the process of respiration and is assumed to be proportional to the oxygen available which is in turn proportional

mammal	body weight (g)	pulse rate
shrew[2]	3.5	782
pipistrelle bat[1]	4	660
bat[2]	6	588
mouse[1]	25	670
hamster[2]	103	347
kitten[2]	117	300
rat[1]	200	420
rat[2]	252	352
guinea pig[1]	300	300
guinea pig[1]	437	269
rabbit[2]	1,340	251
rabbit[1]	2,000	205
oppossum[2]	2,700	187
little dog[1]	5,000	120
seal[2]	22,500	100
big dog[1]	30,000	85
goat[2]	33,000	81
sheep[1]	50,000	70
human[1]	70,000	72
swine[2]	100,000	70
horse[2]	415,000	45
horse[1]	450,000	38
ox[1]	500,000	40
elephant[1]	3,000,000	48

TABLE 2.1: Superscript 1 data source A.J. Clark; superscript 2 data source Altman and Dittmer. See also [1] and [2].

to the blood flow B through the lungs. Hence, $B \propto s$ If we denote the pulse rate as r we may assume

$$B \propto rV$$

where V is the volume of the heart.

We still need to incorporate the body weight w into this model. If we take W to be the weight of the heart assuming constant density of the heart it follows

$$W \propto V$$

Also, if the bodies are assumed to be geometrically similar then $w \propto W$ so by transitivity $w \propto V$ and hence

$$B \propto rw$$

Using the geometric similarity again we can relate the body surface area s to its weight w. From characteristic length scale arguments

$$v^{1/3} \propto s^{1/2}$$

so

$$s \propto w^{2/3}$$

from which we have $rw \propto w^{2/3}$ or

$$r = kw^{-1/3}$$

To validate this model we plot $w^{-1/3}$ versus r for the data Table 2.8. We see that for the larger animals with slower heart rates that this data appears linear and suggests this rather crude model actually is supported by the data. For much smaller animals there appear to be factors that this model is not capturing and the data falls off the line.

2.4 DIMENSIONAL ANALYSIS

In this chapter we have explored modeling with functions and proportion. In some instances, such as the mammalian heart rate, it is possible to cobble enough information together to actually extract a model; in particular, to identify the functional form for the relationship between the dependent and independent variables. Now we turn to a surprisingly powerful and simple tool known as *dimensional analysis*[1].

Dimensional analysis operates on the premise that equations contain terms that have units of measurement and that the validity of these equations, or laws, are not dependent on the system of measurement. Rather these equations relate variables that have inherent physical dimensions that are derived from the fundamental dimensions of *mass, length* and *time*. We label these dimensions generically as M, L and T, respectively.

As we shall see, dimensional analysis provides an effective tool for mathematical modeling in many situations. In particular, some benefits include

[1] This dimension should not be confused with the usual notion of geometric dimension.

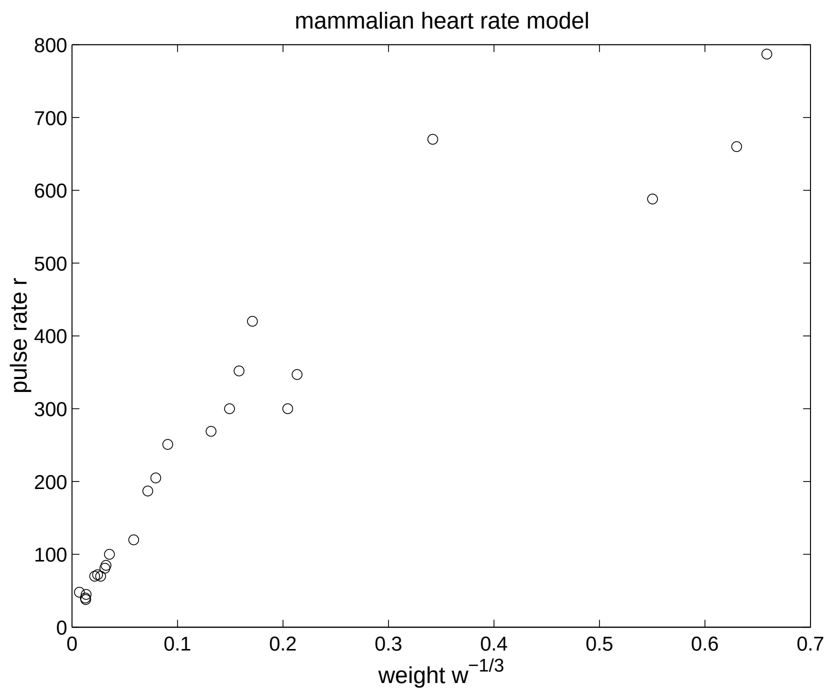


FIGURE 2.8: Testing the model produced by proportionality. For the model to fit, the data should sit on a straight line eminating from the origin.

- determination of the form of a joint proportion
- reduce number of variables in a model
- enforcement of dimensional consistency
- ability to study scaled versions of models

2.4.1 Dimensional homogeneity

An equation is said to be *dimensionally homogeneous* if all the terms in the equation have the same physical dimension.

EXAMPLE 2.12

All the laws of physics are dimensionally homogeneous. Consider Newton's law

$$F = ma$$

The units on the right side are

$$M \cdot \frac{L}{T^2}$$

so we conclude that the physical dimension of a force must be MLT^{-2}. □

EXAMPLE 2.13

The equation of motion of a linear spring with no damping is

$$m\frac{d^2x}{dt^2} + kx = 0$$

What are the units of the spring constant? Dimensionally we can recast this equation as

$$MLT^{-2} + M^aL^bT^cL = 0$$

Matching exponents for each dimension permits the calculation of a, b and c.

$$\begin{aligned} M: &\quad a = 1 \\ L: &\quad 1 = b + 1 \\ T: &\quad -2 = c \end{aligned}$$

Thus we conclude that the spring constant has the dimensions MT^{-2}. □

EXAMPLE 2.14

Let v be velocity, t be time and x be distance. The model equation

$$v^2 = t^2 + \frac{x}{t}$$

is dimensionally inconsistent.

EXAMPLE 2.15

An angle may be defined by the formula

$$\theta = \frac{s}{r}$$

where the arclength s subtends the angle θ and r is the radius of the circle. Clearly this angle is dimensionless.

2.4.2 Discovering Joint Proportions

If in the formulation of a problem we are able to identify a dependent and one or more independent variables, it is often possible to identify the form of a joint proportion. The form of the proportion is actually constrained by the fact that the equations must be dimensionally consistent.

EXAMPLE 2.16 Drag Force on an Airplane

In this problem we consider the drag force F_D on an airplane. As our model we propose that this drag force (dependent variable) is proportional to the independent variables

- cross-sectional area A of airplane
- velocity v of airplane
- density ρ of the air

As a joint proportion we have

$$F_D = kA^a v^b \rho^c$$

where a, b and c are unknown exponents. As a consequence of dimensional consistency we have

$$\begin{aligned} MLT^{-2} &= (M^0L^0T^0)(L^2)^a(\frac{L}{T})^b(\frac{M}{L^3})^c \\ &= M^cL^{2a-3c+b}T^{-b} \end{aligned}$$

From the M exponent we conclude $c = 1$. From the T exponent $b = 2$ and from the L exponent it follows that $1 = 2a - 3c + b$, whence $a = 1$. Thus the only possibility for the form of this joint proportion is

$$F_D = kAv^2\rho$$

Note that if the density of were a constant it would be appropriate to simplify this dependency as

$$F = \tilde{k}Av^2$$

but now the constant $\tilde{k}$ actually has dimensions. □

2.4.3 Procedure for Nondimensionalization

Consider the nonlinear model for a pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

Based on the terms in this model we may express the solution very generally as a relationship between these included terms, i.e.,

$$\phi(\theta, g, l, t) = 0$$

Note that the angle in this model is dimensionless but the other variables all have dimensions. We can convert this equation into a new equation where none of the terms have dimensions. This will be referred to, for obvious reasons, as a dimensional form of the model.

To accomplish this, let

$$\tau = \frac{t}{\sqrt{l/g}}.$$

The substitution of variables may be accomplished by noting that

$$\frac{d^2\theta}{dt^2} = \frac{d^2\theta}{\frac{l}{g}d\tau^2}$$

Thus, after cancelation, the dimensionless form for the nonlinear pendulum model is

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta$$

Now the solution has the general form

$$f(\theta, \tau) = 0,$$

or equivalently,

$$f(\theta, \sqrt{\frac{l}{g}}t) = 0$$

This is a special case of a more general theory.

The Buckingham π-theorem. Any dimensionally homogeneous equation with physical variables $x_1, \ldots, x_m$ expressed

$$\phi(x_1, \ldots, x_m)$$

may be rewritten in terms of its associated dimensionless variables $\pi_1, \ldots, \pi_n$ as

$$f(\pi_1, \ldots, \pi_n) = 0$$

where

$$\pi_k = x_1^{a_{k1}} \ldots x_m^{a_{km}}$$

2.4.4 Modeling with Dimensional Analysis

Now we consider two examples of the application of the ideas described above concerning dimensional analysis. In each of these examples there is more than one dimensionless parameter and it is appropriate to apply the Buckingham π-theorem.

The Pendulum. In this example the goal is to understand how the period of a pendulum depends on the other parameters that describe the nature of the pendulum. The first task is to identify this set of parameters that act as the independent variables on which the period P depends.

Obvious candidates include From this list we are motivated to write

variable	symbol	dimensions
mass	m	M
length	l	L
gravity	g	LT^{-2}
angle	θ_0	$M^0L^0T^0$
period	P	T

TABLE 2.2: Parameters influencing the motion of a simple pendulum.

$$P = \phi(m, l, g, \theta_0)$$

As we shall see, attempting to establish the form of ϕ directly is unnecessarily complicated. Instead, we pursue the idea of dimensional analysis.

To begin this modeling procedure, we compute the values of a, b, c, d and e that make the quantity

$$\pi = m^a l^b g^c \theta_0^d P^e$$

a dimensionless parameter. Again, this is done by equating exponents on the fundamental dimensions

$$M^0L^0T^0 = M^a L^b (LT^{-2})^c (M^0L^0T^0)^d T^e$$

From M^0: $0 = a$.
From L^0: $0 = b + c$.
From T^0: $0 = -2c + e$.
From this we may conclude that

$$\pi = m^0 l^{-c} g^c \theta_0^d P^{2c}$$

or, after collecting terms,

$$\pi = \theta_0^d \left(\frac{gP^2}{l}\right)^c$$

where π is dimensionless for any values of d and c. Thus we have found a complete set of dimensionless parameters

$$\pi_1 = \theta_0$$

and

$$\pi_2 = \sqrt{\frac{g}{l}}P$$

Since the period P of the pendulum is based on dimensionally consistent physical laws we may apply the Buckingham π-theorem. In general,

$$f(\pi_1, \pi_2) = 0$$

which we rewrite as

$$\pi_2 = h(\pi_1)$$

which now becomes

$$P = \sqrt{\frac{l}{g}}h(\theta_0)$$

We may draw two immediate conclusions from this model.

- The period depends on the square root of the length of the pendulum.
- The period is independent of the mass

Of course we have not really shown these conclusions to be "true". But now we have something to look for that can be tested. We could test these assertions and if they contradict our model then we would conclude that we are missing an important factor that governs the period of the pendulum. Indeed, as we have neglected drag forces due to friction it seems our model will have limited validity.

The functional form of h may now be reasonable calculated as there is only one independent variable θ_0. If we select several different initial displacements $\theta_0(i)$ and measure the period for each one we have a set of domain–range values

$$h(\theta_0(i)) = P_i\sqrt{\frac{g}{l}}$$

to which a data fitting procedure may now be applied.

The damped pendulum. We assumed that there was no damping of this pendulum above due to air resistance. We can include a drag force F_D by augmenting the list of relevant parameters to

$$m, l, g, \theta_0, P, F_D$$

Now our dimensionless parameter takes the form

$$\pi = m^a l^b g^c \theta_0^d P^e F_D^f$$

Converting to dimensions

$$M^0L^0T^0 = M^aL^b(LT^{-2})^c(M^0L^0T^0)^dT^e(MLT^{-2})^f$$

As

$$0 = a + f$$

it is no longer possible to immediately conclude that $a = 0$. In fact, it is not. (See problems).

Fluid Flow. Consider the parameters governing the motion of an oil past a spherical ball bearing. Let's assume they include:

variable	symbol	dimensions
velocity	v	LT^{-1}
density	ρ	ML^{-3}
gravity	g	LT^{-2}
radius	l	L
viscosity	μ	$ML^{-1}T^{-1}$

TABLE 2.3: Parameters influencing the motion of a fluid around a submerged body.

The dimensionless combination has the form

$$\pi = v^a \rho^b l^c g^d \mu^e$$

Using the explicit form of the physical dimensions for each term we have

$$M^0 L^0 T^0 = (LT^{-1})^a (ML^{-3})^b (L)^c (LT^{-2})^d (ML^{-1}T^{-1})^e$$

Again, matching exponents

$$\begin{aligned} M: &\quad 0 = b + e \\ L: &\quad 0 = a - 3b + c + d - e \\ T: &\quad 0 = -a - 2d - e \end{aligned}$$

Sinc there are three equations and five unknowns the system is said to be undetermined. Given these numbers, we anticipate that there we can solve for three variable in terms of the other two. Of course, we can solve in terms of *any* of the two variables. For example,

$$\begin{aligned} a &= -2d - e \\ b &= -e \\ c &= d - e \end{aligned}$$

Plugging these constraints into our expression for π gives

$$\pi = \left(\frac{v^2}{lg}\right)^{-d} \left(\frac{\rho l v}{\mu}\right)^{-e}$$

Thus, our two dimensionless parameters are the *Froude number*

$$\pi_1 = \frac{v^2}{lg}$$

and the *Reynolds number*

$$\pi_2 = \frac{v \rho l}{\mu}$$

For further discussion see Giordano, Wells and Wilde, UMAP module 526.

PROBLEMS

2.1. By drawing a new graph, show the effect of the size of the island on the

- extinction curve
- migration curve

Now predict how island size impacts the number of species on the island. Does this seem reasonable?

2.2. Give an example of a commodity that does not obey the

- law of supply
- law of demand

and justify your claim.

2.3. Translate into words the qualitative interpretation of the slope of the supply and demand curves. In particular, what is the meaning of a

- flat supply curve?
- steep supply curve?
- steep demand curve?

2.4. Consider the table of market adjustments below. Assuming the first point is on the demand curve, compute the equations of both the demand and supply curve. Using these equations, find the missing values A, B, C, D. What is the equilibrium point? Do you think the market will adjust to it?

quantity	price
3	0.7
0.14	0.7
0.14	0.986
0.1972	0.986
$A =?$	$B =?$
$C =?$	$D =?$

2.5. Using the cobweb plot show an example of a market adjustment that oscillates wildly out of control. Can you describe a qualitative feature of the supply and demand curves that will ensure convergence to an equilibrium?

2.6. Consider the effect of a price increase on airplane fuel (kerosene) on the airline industry. What effect does this have on the supply curve? Will the airline industry be able to pass this cost onto the flying public? How does your answer differ if the demand curve is flat versus steep?

2.7. Prove properties 2.1 and 2.2.

2.8. Is the temperature measured in degrees Fahrenheit proportional to the temperature measured in degrees centigrade?

2.9. Consider the Example 2.6 again. Demonstrate the proportionalities stated. For the case of the breaking distance equate the work done by the breaks to the dissipated kinetic energy of the car.

2.10. Items at the grocery store typically come in various sizes and the cost per unit is generally smaller for larger items. Model the cost per unit weight by considering the superposition of proportions due to the costs of

- production
- packaging

- shipping

the product. What predictions can you make from this model. This problem was adapted from Bender [1].

2.11. Go to your nearest supermarket and collect data on the cost of items as a function of size. Do these data behave in a fashion predicted by your model in the previous problem?

2.12. In this problem take the diagonal of a rectangle as it's length scale l. Show by direct calculation that this can be used to measure the area, i.e.,

$$A = \alpha l^2$$

Determine the constant of proportionality α in terms of the shape factor of the rectangle.

2.13. Consider a radiator designed as a spherical shell. If the characteristic length of the shell doubles (assume the larger radiator is geometrically similar to the smaller radiator) what is the effect on the amount of heat loss? What if the design of the radiator is a parallelpiped instead?

2.14. How does the argument in Example 2.10 change if the falling object is not spherical but some other irregular shape?

2.15. Extend the definition of geometric similarity for

- parallelpipeds
- irregularly shaped objects

Can you propose a computer algorithm for testing whether two objects are geomtrically similar?

2.16. Consider the force on a pendulum due to air friction modeled by

$$F_D = \kappa v^2$$

Determine the units of κ.

2.17. Newton's law of gravitation states that

$$F = \frac{G m_1 m_2}{r^2}$$

where F is the force between two objects of masses m_1, m_2 and r is the distance between them.

(a) What is the physical dimension of G?

(b) Compute two dimensionless products π_1 and π_2 and show explicitly that they satisfy the Buckingham π-theorem.

2.18. This problem concerns the pendulum example described in subsection 2.4.4. Repeat the analysis to determine the dimensionless parameter(s) but now omit the gravity term g. Discuss.

2.19. This problem concerns the pendulum example described in subsection 2.4.4. Repeat the analysis for determining all the dimensionless parameters but now include a parameter κ associated with the drag force of the form $F_D = \kappa v$. Hint: first compute the dimensions of κ.

2.20. Convert the equation governing the distance travelled by a projectile,

$$\frac{d^2 x}{dt^2} = \frac{-gR^2}{(x+R)^2},$$

to the form

$$\frac{d^2y}{d\tau^2} = \frac{-1}{(y+1)^2},$$

where y and τ are dimensionless.

2.21. Reconsider the example in subsection 2.4.4. Instead of solving for a, b, c in terms of d, e solve for c, d, e in terms of a and b. Show that now

$$\pi'_1 = \frac{v}{\sqrt{lg}}$$

and

$$\pi'_2 = \frac{\rho l^{3/2} g^{1/2}}{\mu}$$

Show also that both π'_1 and π'_2 can be written in terms of π_1 and π_2.

2.22. Consider an object with surface area A traveling with a velocity v through a medium with kinematic viscosity μ and density ρ.

(a) Assuming the effect of μ is small compute the drag force due to the density F_ρ.

(b) Assuming the effect of ρ is small compute the drag force due to the kinematic viscosity F_μ.

c) Compute the dimensionless ratio of these drag forces and discuss what predictions you can make.

2.23. Assume a drag force of the form

$$F_d = \kappa v^2$$

acts on a pendulum in addition to the gravity force. Use dimensional analysis to show that the solution of the pendulum equation can be written in the form

$$\theta = \psi(t\sqrt{l/g}, l\kappa/m).$$

2.24. How does the required power P of a helicopter engine depend on the length of the rotors l? The rotors are pushing air so presumable the density ρ as well as the weight of the helicopter $w = mg$ are variables that affect the power requirement. Draw a sketch of your result plotting P versus l. See [3] for more discussion of this problem.

Bibliography

1. Edward A. Bender. *An Introduction to Mathematical Modeling.* John Wiley & Sons, New York, 1978.
2. F.R. Giordano, M.D. Weir, and W.P. Fox. *A First Course in Mathematical Modeling.* Brooks/Cole, Pacific Grove, 1997.
3. T.W. Körner. *The Pleasures of Counting.* Cambridge University Press, Cambridge, U.K., 1996.
4. C.C. Lin and L.A. Segel. *Mathematics Applied to Deterministic Problems in the Natural Sciences.* Macmillan, New York, 1974.