Stability of Equilibria of Difference Equations

Consider the difference equation

$$x_{n+1} = f(x_n) \tag{1}$$

for a real variable x_n, $n = 0, 1, 2, \ldots$. Assume $\overline{x}$ is an equilibrium point, that is, a solution of the equation

$$\overline{x} = f(\overline{x}). \tag{2}$$

Definition An equilibrium $\overline{x}$ of the difference equations (1) is asymptotically stable, if for any initial condition x_0 close to $\overline{x}$ the iterates x_n remain close to $\overline{x}$ for all $n > 0$, and $x_n \to \overline{x}$ as $n \to \infty$. The equilibrium $\overline{x}$ is unstable, if there exists a constant $c > 0$ such that for any initial condition that is arbitrarily close to $\overline{x}$ yet $|x_0 - \overline{x}| \neq 0$, there is an $n^* > 0$ such that $|x_{n^*} - \overline{x}| > c$.

To study the stability of $\overline{x}$, consider perturbations

$$\xi_n = x_n - \overline{x}, \quad \text{that is,} \quad x_n = \overline{x} + \xi_n.$$

A difference equation for ξ_n is found as

$$\xi_{n+1} = x_{n+1} - \overline{x} = f(x_n) - \overline{x} = f(\overline{x} + \xi_n) - \overline{x}. \tag{3}$$

If $|\xi_n|$ is small, we can make a Taylor expansion,

$$f(\overline{x} + \xi_n) - \overline{x} = f'(\overline{x})\xi_n + \frac{1}{2} f''(\overline{x})\xi_n^2 + \cdots . \tag{4}$$

Substituting (4) into (3) and keeping only the linear term (proportional to ξ_n) gives the so called *linearized difference equation*,

$$\xi_{n+1} = f'(\overline{x})\xi_n. \tag{5}$$

Since $f'(\overline{x})$ is just a real number we know that the solution of (5) is simply given by

$$\xi_n = \left(f'(\overline{x})\right)^n \xi_0.$$

Thus $\xi_n \to 0$ ($n \to \infty$) if $|f'(\overline{x})| < 1$ and $|\xi_n| \to \infty$ if $|f'(\overline{x})| > 1$, suggesting that the equilibrium $\overline{x}$ is asymptotically stable and unstable in the former and latter cases, respectively. This is indeed the case:

Theorem If $|f'(\overline{x})| < 1$ then $\overline{x}$ is asymptotically stable, and if $|f'(\overline{x})| > 1$ then $\overline{x}$ is unstable.

Note that the theorem does not give any clue about the stability of $\overline{x}$ if $|f'(\overline{x})| = 1$.

Example 1

$$\begin{aligned} p_{n+1} &= p_n - \frac{1}{10} p_n (1-p_n)(2-p_n) \\ &= \frac{4}{5} p_n + \frac{3}{10} p_n^2 - \frac{1}{10} p_n^3 \equiv f(p_n). \end{aligned}$$

The equilibria are determined by $p_{n+1} = p_n \equiv \overline{p}$ and are given by

$$\overline{p}_1 = 0, \quad \overline{p}_2 = 1, \quad \overline{p}_3 = 2.$$

We calculate the derivatives of f at these equilibrium points to determine their stability:

$$\begin{aligned} f'(0) &= \tfrac{4}{5} < 1 && \Rightarrow \ \overline{p}_1 \text{ is asymptotically stable} \\ f'(1) &= \tfrac{4}{5} + \tfrac{3}{5} - \tfrac{3}{10} = \tfrac{11}{10} > 1 && \Rightarrow \ \overline{p}_2 \text{ is unstable} \\ f'(2) &= \tfrac{4}{5} + \tfrac{3}{5} \cdot 2 - \tfrac{3}{10} \cdot 4 = \tfrac{4}{5} < 1 && \Rightarrow \ \overline{p}_3 \text{ is asymptotically stable.} \end{aligned}$$

Example 2

$$p_{n+1} = r p_n - r p_n^2 \equiv f(p_n),$$

where r is a parameter with $r > 0$. Since p_n models a population, we require that p_n be nonnegative. The equilibria $\overline{p}$ satisfy $\overline{p} = r\overline{p} - \overline{p}^2$. There are two solutions,

$$\overline{p}_1 = 0 \ \text{ and } \ \overline{p}_2 \ \text{ satisfies } \ 1 = r - r\overline{p}_2 \ \Rightarrow \ \overline{p}_2 = 1 - \frac{1}{r} \ \text{ for } \ r \geq 1,$$

since $p \geq 0$. We calculate again derivatives to determine for which values of r the equilibria are asymptotically stable or unstable:

$$\begin{aligned} f'(0) &= r \\ &\Rightarrow \ \overline{p}_1 \ \text{ is asymptotically stable for } \ 0 < r < 1 \ \text{ and unstable for } \ r > 1 \\ f'(1-1/r) &= r - 2r(1-1/r) = 2 - r \\ &\Rightarrow \ \overline{p}_2 \ \text{ is asymptotically stable for } \ 1 < r < 3 \ \text{ and unstable for } \ r > 3. \end{aligned}$$

Note that $f'(0) = 1$ for $r = 1$ as well as $f'(1-1/r) = 1$ for $r = 1$, while $f'(1-1/r) = -1$ when $r = 3$. Thus when r increases through the special parameter values $r = 1$ and $r = 3$ the stability properties of the equilibrium points change (and at $r = 1$ a new equilibrium, namely $\overline{p}_2$, is born and coincides at this parameter value with $\overline{p}_1$). Parameter values of this kind are called *bifurcation points*.