9.7: Qualitative (Stability) Analysis

$$\mathbf{x}' = A\mathbf{x}, \quad A : n \times n \qquad (1)$$

$\mathbf{x}(t) = \mathbf{0}$ is equilibrium solution

Consider general system:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \qquad (2)$$

Assume equilibrium $\mathbf{x}(t) = \mathbf{x}_0$:

$$\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$$

Def.:

- $\mathbf{x}_0$ is stable if for any $\epsilon > 0$ there is a $\delta > 0$ s.t. $|\mathbf{x}(t) - \mathbf{x}_0| < \epsilon$ for all $t > 0$ whenever $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$. (Solutions that start close to $\mathbf{x}_0$ remain close.)
- $\mathbf{x}_0$ is unstable if it is not stable. (There are solutions starting arbitrarily close to $\mathbf{x}_0$ that move 'far away' from $\mathbf{x}_0$.)
- $\mathbf{x}_0$ is asymptotically stable if $\mathbf{x}_0$ is stable and there is $\eta > 0$ s.t. $\mathbf{x}(t) \to \mathbf{x}_0$ for $t \to \infty$ whenever $|\mathbf{x}(0) - \mathbf{x}_0| < \eta$.

Def.:

- An asymptotically stable equilibrium $\mathbf{x}_0$ of (2) is a sink.
- An equilibrium $\mathbf{x}_0$ of (2) is a source if every solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0|$ arbitrarily small eventually moves 'far away' from $\mathbf{x}_0$ when t increases.

Examples:

Let A be 2×2.

The equilibrium $\mathbf{x}_0 = \mathbf{0}$ of (1) is

- a sink if the phase portrait is a nodal or spiral sink
- a source if the phase portrait is a nodal or spiral source
- unstable if the phase portrait is a saddle
- stable but not asymptotically stable if the phase portrait is a center or stable saddle-node.

Thm.: Let A be $n \times n$

1. If $\mathrm{Re}(\lambda) < 0$ for all eigenvalues of A ($\lambda < 0$ if λ is real), then $\mathbf{x}(t) \to \mathbf{0}$ for $t \to \infty$ for any solution $\mathbf{x}(t)$ of (1).
   ($\mathbf{0}$ is a sink)
2. If there is an eigenvalue λ of A with $\mathrm{Re}(\lambda) > 0$ ($\lambda > 0$ if λ is real), then there are solutions $\mathbf{x}(t)$ of (1) with $|\mathbf{x}(0)|$ arbitrarily small that get arbitrarily large when t increases.
   ($\mathbf{0}$ is unstable)
3. If $\mathrm{Re}(\lambda) > 0$ for all eigenvalues λ of A, then every solution $\mathbf{x}(t)$ of (1) with $\mathbf{x}(0) \neq \mathbf{0}$ gets arbitrarily large when t increases.
   ($\mathbf{0}$ is a source)
4. If $\mathrm{Re}(\lambda) \leq 0$ for all eigenvalues λ of A, and for any eigenvalue with $\mathrm{Re}(\lambda) = 0$ every generalized eigenvector is an eigenvector, then **$\mathbf{0}$ is stable**. *(Ex.: stable saddle-nodes, centers)*

Proof 1. Every entry of $\mathbf{x}(t)$ consists of terms of the form $e^{\lambda t} t^k$ or $e^{\alpha t} t^k \cos \beta t$, $e^{\alpha t} t^k \sin \beta t$, with nonnegative integers k and $\lambda < 0$, $\alpha < 0$. All these terms decay to 0 if $t \to \infty$.

For $\mathbf{n = 2}$:

- $D > 0$, $T < 0 \Rightarrow$ sink
- $D > 0$, $T > 0 \Rightarrow$ source
- $D < 0 \Rightarrow$ saddle $\Rightarrow$ unstable but not source

Ex.: $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} x_0 \\ e^{-t} y_0 \end{bmatrix}$

$\lambda = 0 \leftrightarrow \mathbf{v} = [1, 0]^T$: $\mathbf{0}$ is stable

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Ex.: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} T = 0 \\ D = 0 \end{Bmatrix} \Rightarrow p(\lambda) = \lambda^2$

$\lambda = 0 \leftrightarrow \mathbf{v} = [1, 0]^T$

$A^2 = 0 \Rightarrow \begin{cases} \text{every vector is} \\ \text{generalized eigenvector} \end{cases}$

Solution:

$$\mathbf{x}(t) = (I + At)\mathbf{x}_0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 + t x_0 \end{bmatrix}$$

$\Rightarrow \mathbf{0}$ is unstable (but not a source)

Worked Out Examples from Exercises

Ex. 1: Classify $\mathbf{0}$ as unstable equilibrium, stable equilibrium, sink or source of $\mathbf{x}' = A\mathbf{x}$ for the given A. Verify the classification through a phase portrait.

$$A = \begin{bmatrix} -0.2 & 2 \\ -2 & -0.2 \end{bmatrix}: \; D = 4.04 > 0, \; T = -0.4 < 0 \Rightarrow \text{sink (spiral sink)}$$

Ex. 3: Same as Ex. 1 for $A = \begin{bmatrix} -6 & -15 \\ 3 & 6 \end{bmatrix}$.

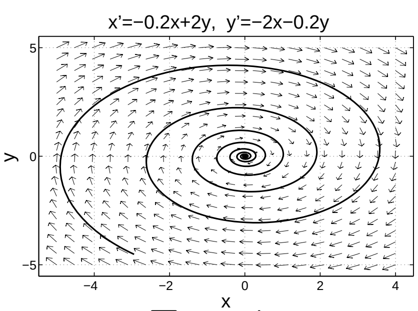
$D = 9$, $T = 0 \Rightarrow$ center $\Rightarrow$ stable (but not sink)

Ex. 5: Same as Ex. 1 for $A = \begin{bmatrix} 0.1 & 2 \\ -2 & 0.1 \end{bmatrix}$.

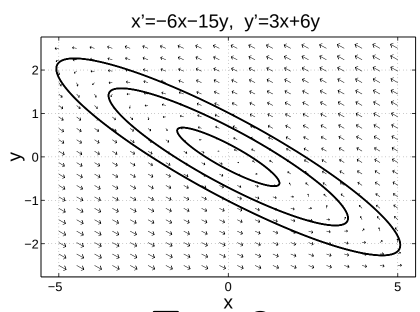
$D = 4.04$, $T = 0.2 \Rightarrow$ source (phase portrait: spiral source)

Ex. 7: Same as Ex. 1 for $A = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}$.

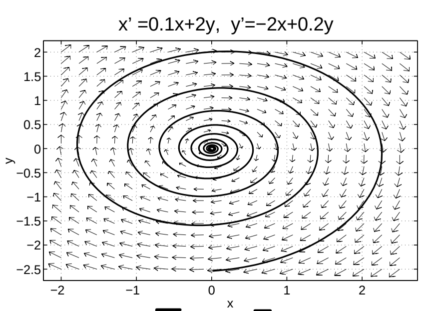
$D = -2 \Rightarrow$ saddle $\Rightarrow$ unstable (but not source)



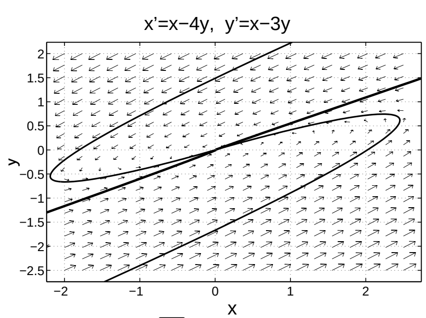
Ex. 1



Ex. 3



Ex. 5



Ex. 7

Ex. 9: Classify $\mathbf{0}$ as unstable equilibrium, stable equilibrium, sink or source of $\mathbf{x}' = A\mathbf{x}$ for the given A. Verify the classification by plotting some solutions.

$$A = \begin{bmatrix} -3 & -4 & 2 \\ -2 & -7 & 4 \\ -3 & -8 & 4 \end{bmatrix}$$

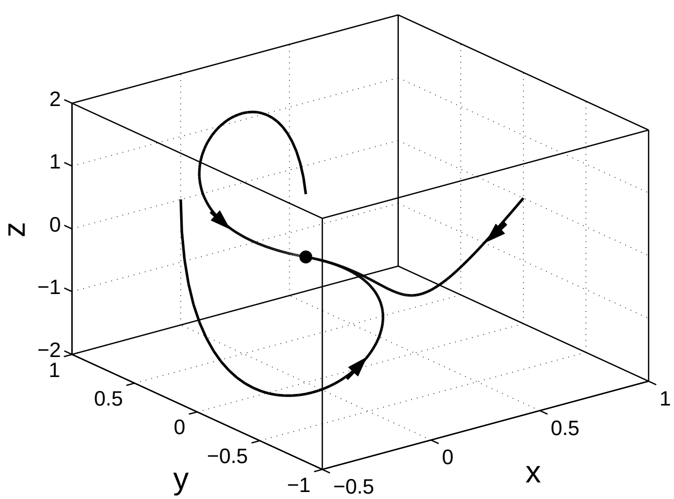
Matrix is a bit complicated, so use Matlab's *eig* command to find eigenvalues.
Result: $\lambda_1 = -3 < 0$, $\lambda_2 = -1 < 0$, $\lambda_3 = -2 < 0 \Rightarrow$ **sink**

To plot trajectories in Matlab, generate solution arrays using the *expm* command and plot them using the *plot3* command.
Chosen initial values: $[1,0,0]^T$, $[0,1,0]^T$, $[0,0,1]^T$.

Find eigenvalues:

```
>> A=[-3 -4 2;-2 -7 4;-3 -8 4];
>> eig(A)'
ans =
   -3.0000   -1.0000   -2.0000
```

Plot after rotating and editing:



Generate solution arrays and plot them together with equilibrium point $\mathbf{0}$: (commands executed in script file)

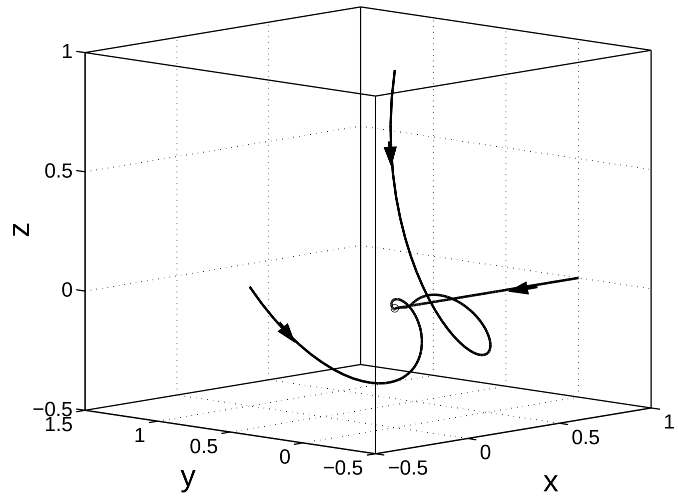
```
A=[-3 -4 2;-2 -7 4;-3 -8 4];
M=200;t=linspace(0,6,M);
x01=[1;0;0];x02=[0;1;0];x03=[0;0;01];
for n=1:M
    x1(:,n)=expm(A*t(n))*x01;
    x2(:,n)=expm(A*t(n))*x02;
    x3(:,n)=expm(A*t(n))*x03;
end
plot3(x1(1,:),x1(2,:),x1(3,:),'k',...
   x2(1,:),x2(2,:),x2(3,:),'k',...
   x3(1,:),x3(2,:),x3(3,:),'k',...
   0,0,0,'ko'),
xlabel('x'),ylabel('y'),zlabel('z')
```

Ex. 11: Same as Ex. 9 for $A = \begin{bmatrix} -1 & 3 & 1 \\ 0 & 1 & 6 \\ 0 & -3 & -5 \end{bmatrix}$

No problem to find eigenvalues analytically:

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} -1-\lambda & 3 & 1 \\ 0 & 1-\lambda & 6 \\ 0 & -3 & -5-\lambda \end{vmatrix} \\ &= (-1)^{1+1}(-1-\lambda) \begin{vmatrix} 1-\lambda & 6 \\ -3 & -5-\lambda \end{vmatrix} \\ &= -(\lambda+1)[(1-\lambda)(-5-\lambda)+18] \\ &= -(\lambda+1)(\lambda^2+4\lambda+13) \\ &= -(\lambda+1)[(\lambda+2)^2+9] \\ &= -(\lambda+1)(\lambda+2-3i)(\lambda+2+3i) \end{aligned}$$

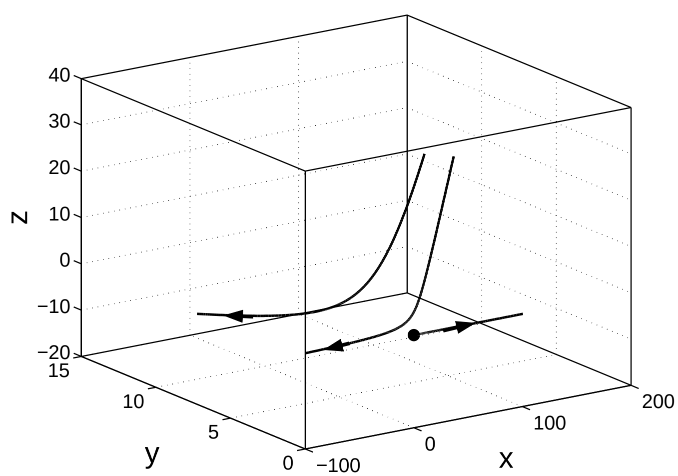
$\Rightarrow$ eigenvalues $-1, -2+3i, -2-3i$
$\Rightarrow$ **sink**



Ex. 14: As Ex. 9 for $A = \begin{bmatrix} 3 & -3 & -5 \\ 0 & 1 & 0 \\ 0 & -3 & -2 \end{bmatrix}$

$$\begin{aligned} p(\lambda) &= (-1)^{2+2}(1-\lambda) \begin{vmatrix} 3-\lambda & -5 \\ 0 & -2-\lambda \end{vmatrix} \\ &= -(\lambda-1)(\lambda-3)(\lambda+2) \end{aligned}$$

$\Rightarrow$ eigenvalues $1, 3, -2 \Rightarrow$ **unstable**
(no source because of -2)



Ex. 15: Same as Ex. 9 for the given 4×4-matrix A

$$A = \begin{bmatrix} 3 & -2 & -5 & 3 \\ 16 & -6 & -17 & 9 \\ -14 & 5 & 15 & -8 \\ -19 & 8 & 23 & -13 \end{bmatrix}$$

Matlab's *eig* command $\Rightarrow$ eigenvalues $2, -1, -1, -1$ (triple eigenvalue -1) $\Rightarrow$ **unstable** (no source)

Since $4d$ trajectories cannot be displayed, plot x_i versus t for two ICs:

```
A=[3 -2 -5 3;16 -6 -17 9;-14 5 15 -8;-19 8 23 -13];
M=200;t=linspace(0,1,M);
x01=[0.1;0.1;0.1;0.05];x02=[-0.01;-0.01;-0.01;0.001];
for n=1:M
    x1(:,n)=expm(A*t(n))*x01;
    x2(:,n)=expm(A*t(n))*x02;
end
figure(1),plot(t,x1(1,:),'k',t,x1(2,:),'k',t,x1(3,:),'k',t,x1(4,:),'k'),
xlabel('t'),ylabel('x_i(t)'),title('IC: [0.1,0.1,0.1,0.05]^T')
figure(2),plot(t,x2(1,:),'k',t,x2(2,:),'k',t,x2(3,:),'k',t,x2(4,:),'k')
xlabel('t'),ylabel('x_i(t)'),title('IC: [-0.01,-0.01,-0.01,0.001]^T')
```