9.6: Matrix Exponential, Repeated Eigenvalues

$$\mathbf{x}' = A\mathbf{x}, \quad A : n \times n \qquad (1)$$

Def.: If $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ is a fundamental set of solutions (F.S.S.) of (1), then

$$X(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)] \ (n \times n)$$

is called a fundamental matrix (F.M.) for (1).

General solution:
$(\mathbf{c} = [c_1, \ldots, c_n]^T)$

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) \\ &= X(t)\mathbf{c} \end{aligned}$$

Thm.: If $X(t)$ is a F.M. for (1) and C is a constant nonsingular matrix, then $X(t)C$ is also a F.M.

Proof: Each column of $X(t)C$ is a linear combination of the columns of $X(t)$ and so is a solution of (1), and $X(0)C$ is nonsingular.

Ex.: $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1 \quad \leftrightarrow \quad \mathbf{v}_1 = [2, 3]^T$$
$$\lambda_2 = -2 \quad \leftrightarrow \quad \mathbf{v}_2 = [1, 1]^T$$

F.S.S.:

$$\mathbf{x}_1(t) = e^{-t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{x}_2(t) = e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

F.M.: $\quad X(t) = \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix}$

If we set

$$\mathbf{y}_1(t) = 2\mathbf{x}_2(t), \ \ \mathbf{y}_2(t) = 3\mathbf{x}_2(t),$$

$\mathbf{y}_1(t), \mathbf{y}_2(t)$ are also F.S.S. with F.M.

$$\begin{aligned} Y(t) &= \begin{bmatrix} 3e^{-2t} & 4e^{-t} \\ 3e^{-2t} & 6e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \end{aligned}$$

Matrix Exponential

Consider IVP:

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0 \qquad (2)$$

Solution of IVP: If $X(t)$ is a F.M., the general solution is

$$\mathbf{x}(t) = X(t)\mathbf{c}$$

Match $\mathbf{c}$ to IC:

$$\begin{aligned} \mathbf{x}(0) &= X(0)\mathbf{c} = \mathbf{x}_0 \\ \Rightarrow \quad \mathbf{c} &= (X(0))^{-1}\mathbf{x}_0 \\ \Rightarrow \quad \mathbf{x}(t) &= X(t)(X(0))^{-1}\mathbf{x}_0 \end{aligned}$$

Def.: Given a F.M. $X(t)$, then

$$e^{At} \stackrel{def}{=} X(t)(X(0))^{-1}$$

is the matrix exponential of At.

Thm.: The solution of (2) is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

Ex.: $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$

F.M.: $X(t) = \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix}$

$X(0) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$, $(X(0))^{-1} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

$$\begin{aligned} e^{At} &= X(t)(X(0))^{-1} \\ &= \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-2t} - 2e^{-t} & 2e^{-t} - 2e^{-2t} \\ 3e^{-2t} - 3e^{-t} & 3e^{-t} - 2e^{-2t} \end{bmatrix} \end{aligned}$$

IVP: $\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solution:

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 3e^{-2t} - 2e^{-t} & 2e^{-t} - 2e^{-2t} \\ 3e^{-2t} - 3e^{-t} & 3e^{-t} - 2e^{-2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-2t} - 2e^{-t} \\ 4e^{-2t} - 3e^{-t} \end{bmatrix} \end{aligned}$$

Properties of the Matrix Exponential

- Exponential series ($A^0 = I$):

$$e^{At} = \sum_{m=0}^{\infty} (At)^m / m!$$

Convergence for any matrix A

- $$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$$

- If $D = [d_{ij}]$ is a diagonal matrix ($d_{ij} = 0$ for $i \neq j$), then e^{Dt} is a diagonal matrix with entries $e^{d_{ii}t}$. Ex.:

$$\exp\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} t \right) = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

- Special case ($d_{ii} = r$):

$$e^{(rI)t} = e^{rt} I$$

- If $AB = BA$, then

$$e^{(A+B)t} = e^{At} e^{Bt} = e^{Bt} e^{At}$$

Note: If $AB \neq BA$, then in general

$$e^{(A+B)t} \neq e^{At} e^{Bt} \neq e^{Bt} e^{At}$$

- e^{At} is nonsingular, and

$$(e^{At})^{-1} = e^{-At}$$

- If V is nonsingular, then

$$e^{(VAV^{-1})t} = V e^{At} V^{-1}$$

- If $\mathbf{v}$ is an eigenvector for an eigenvalue λ, then

$$e^{At} \mathbf{v} = e^{\lambda t} \mathbf{v}$$

Matrices with only one eigenvalue

Thm.: If A has only one eigenvalue λ, then there is an integer k, $0 < k \leq n$, such that

$$(A - \lambda I)^k = 0$$

Use this to compute e^{At} as follows. Write $A = \lambda I + (A - \lambda I)$. Then

$$\begin{aligned} e^{At} &= e^{(\lambda I)t + (A-\lambda I)t} \\ &= e^{(\lambda I)t} e^{(A-\lambda I)t} \\ &= e^{\lambda t} e^{(A-\lambda I)t} \\ &= e^{\lambda t} \sum_{j=0}^{k-1} (A - \lambda I)^j (t^j / j!) \end{aligned}$$

$\Rightarrow$ only k terms of exponential series required

Ex.: $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$: $T = -2$, $D = 1$

$$\begin{aligned} p(\lambda) &= \lambda^2 - T\lambda + D \\ &= \lambda^2 + 2\lambda + 1 \\ &= (\lambda + 1)^2 \end{aligned}$$

$\Rightarrow$ only one eigenvalue $\lambda = -1$

$$\begin{aligned} A + I &= \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \\ (A + I)^2 &= \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (k = 2) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} e^{(A+I)t} &= I + (A + I)t \\ &= \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix} \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} e^{At} &= e^{-t} e^{(A+I)t} \\ &= e^{-t} \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix} \end{aligned}$$

Generalized Eigenvectors and Associated Solutions

If A has repeated eigenvalues, n linearly independent eigenvectors may not exist → need generalized eigenvectors

Def.: Let λ be eigenvalue of A.

(a) The algebraic multiplicity, m, of λ is the multiplicity of λ as root of the characteristic polynomial (CN Sec. 9.5).

(b) The geometric multiplicity, m_g, of λ is $\dim \text{null}(A - \lambda I)$.

Need: m linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$ associated with λ.

- If $m_g = m \Rightarrow m$ linearly independent eigenvector solutions.
- What if $m_g < m$?

Thm.: If λ is an eigenvalue with algebraic multiplicity m, then there is an integer k, $0 < k \leq m$, such that

$$\begin{aligned} \dim \text{null}((A - \lambda I)^k) &= m \\ \dim \text{null}((A - \lambda I)^{k-1}) &< m \end{aligned}$$

Def.: Any nonzero vector $\mathbf{v}$ in $\text{null}((A - \lambda I)^k)$ is a generalized eigenvector for λ.

<u>*Solution associated with $\mathbf{v}$:*</u>

$$(A - \lambda I)^k \mathbf{v} = \mathbf{0} \quad \Rightarrow$$

$$e^{At}\mathbf{v} = e^{\lambda t} \sum_{j=0}^{k-1} (t^j/j!)(A - \lambda I)^j \mathbf{v}$$

Thm.: Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be a basis of $\text{null}(A - \lambda I)^k$. Then the

$$\mathbf{x}_i(t) = e^{\lambda t} \sum_{j=0}^{k-1} (t^j/j!)(A - \lambda I)^j \mathbf{v}_i,$$

$1 \leq i \leq m$, are m linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$.

2d Systems: (Sec. 9.2)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \left\{ \begin{array}{rcl} T & = & a+d \\ D & = & ad-bc \end{array} \right\}$$

Assume $T^2 - 4D = 0 \Rightarrow$

$p(\lambda) = (\lambda - \lambda_1)^2, \ \ \lambda_1 = T/2$

(a) If $A = \lambda_1 I \Rightarrow m_g = 2$

$\Rightarrow \ \mathbf{x}(t) = e^{\lambda_1 t}\mathbf{x}(0)$

(any vector is eigenvector)

(b) If $A \neq \lambda_1 I \Rightarrow m_g = 1$:

- Compute eigenvector $\mathbf{v}$
- Pick vector $\mathbf{w}$ that is *not* a multiple of $\mathbf{v}$

  $\Rightarrow (A - \lambda_1 I)\mathbf{w} = a\mathbf{v}$

  for some $a \neq 0$ (any $\mathbf{w} \in \mathbf{R}^2$ is generalized eigenvector)
- $\Rightarrow$ F.S.S.:

$$\begin{array}{rcl} \mathbf{x}_1(t) & = & e^{\lambda_1 t}\mathbf{v} \\ \mathbf{x}_2(t) & = & e^{\lambda_1 t}(\mathbf{w} + a\mathbf{v}t) \end{array}$$

Ex.: $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$: $T = -2$, $D = 1$

$\Rightarrow T^2 - 4D = 0 \Rightarrow$ eigenvalue $\lambda = -1$

$$A + I = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

$\Rightarrow$ eigenvector $\mathbf{v} = [-2, 1]^T$

Choose $\mathbf{w} = [1, 0]^T$ (simple form) $\Rightarrow$

$$(A+I)\mathbf{w} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\mathbf{v}$$

$\Rightarrow$ F.S.S.:

$$\begin{array}{rcl} \mathbf{x}_1(t) & = & e^{-t}\mathbf{v} = e^{-t}[-2, 1]^T \\ \mathbf{x}_2(t) & = & e^{-t}(\mathbf{w} - \mathbf{v}t) \\ & = & e^{-t}([1, 0]^T - t[-2, 1]^T) \\ & = & e^{-t}[1 + 2t, -t]^T \end{array}$$

Other Method: Compute (c.f. p.4)

$$e^{At} = e^{-t}(I + (A+I)t) = e^{-t} \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix}$$

Columns of e^{At} are also F.S.S.

Ex.: $A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -3 \end{bmatrix}$

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} -1-\lambda & 2 & 1 \\ 0 & -1-\lambda & 0 \\ -1 & -3 & -3-\lambda \end{vmatrix} \\ &= (-1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ -1 & -3-\lambda \end{vmatrix} \\ &= (-1-\lambda)[(1+\lambda)(3+\lambda)+1] \\ &= -(\lambda+1)(\lambda^2+4\lambda+4) \\ &= -(\lambda+1)(\lambda+2)^2 \end{aligned}$$

$\Rightarrow$ eigenvalues $\lambda_1 = -1$, $m_1 = 1$
$\lambda_2 = -2$, $m_2 = 2$

Compute $A - \lambda_1 I = A + I$:

$$A+I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Set $x_3 = -2$
$\Rightarrow$ eigenvector $\mathbf{v}_1 = [1, 1, -2]^T$

Since $m_1 = 1$
$\Rightarrow$ one (eigenvector) solution:

$$\mathbf{x}_1(t) = e^{-t}[1, 1, -2]^T$$

$m_2 = 2 \rightarrow$ check $A - \lambda_2 I$, $(A - \lambda_2 I)^2$:

$$A+2I = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A+2I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A + 2I \rightarrow$ eigenvector $\mathbf{v}_2 = [1, 0, -1]^T$
$\Rightarrow$ eigenvector solution:

$$\mathbf{x}_2(t) = e^{-2t}[1, 0, -1]^T$$

For $\mathbf{x}_3(t)$ use generalized eigenvector $\mathbf{v}_3$ that is *not* an eigenvector.

Basis of null$((A+2I)^2)$: $\begin{cases} \mathbf{u}_1 = [1, 0, 0]^T \\ \mathbf{u}_2 = [0, 0, 1]^T \end{cases}$

Note: $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2$
$= (A + 2I)\mathbf{u}_1 = (A + 2I)\mathbf{u}_2$

$\mathbf{v}_3$ can be any vector $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ s.t.
$(A+2I)(c_1\mathbf{u}_1+c_2\mathbf{u}_2) = (c_1+c_2)\mathbf{v}_2 \neq \mathbf{0}$
Choose $\mathbf{v}_3 = \mathbf{u}_2 = [0, 0, 1]^T$ (text: $\mathbf{u}_1$)

$$\begin{aligned} \Rightarrow \quad \mathbf{x}_3(t) &= e^{-2t}(I\mathbf{v}_3 + t(A+2I)\mathbf{v}_3) \\ &= e^{-2t}(\mathbf{v}_3 + t\mathbf{v}_2) \\ &= e^{-2t}[t, 0, 1-t]^T \end{aligned}$$

Ex.: $A = \begin{bmatrix} 6 & 6 & -3 & 2 \\ -4 & -4 & 2 & 0 \\ 8 & 7 & -4 & 4 \\ 1 & 0 & -1 & -2 \end{bmatrix}$

Matlab $\to p(\lambda) = ((\lambda+1)^2+1)^2$

$\Rightarrow$ single complex pair of eigenvalues

$\lambda_1 = -1+i$, $\lambda_2 = \overline{\lambda_1}$ $(m=2)$.

---

1. Check $B \equiv A - \lambda_1 I = A - (-1+i)I$:

$$B = \begin{bmatrix} 7-i & 6 & -3 & 2 \\ -4 & -3-i & 2 & 0 \\ 8 & 7 & -3-i & 4 \\ 1 & 0 & -1 & -1-i \end{bmatrix}$$

Matlab $\to$ basis for null(B):

$$\mathbf{v}_1 = [2, 0, 4, -1+i]^T$$

$\Rightarrow$ Complex eigenvector solution:

$$\mathbf{z}_1(t) = e^{(-1+i)t}[2, 0, 4, -1+i]^T$$

2. Check $B^2 = (A - \lambda_1 I)^2$:

$$B^2 = \begin{bmatrix} 2-14i & 3-12i & -2+6i & -4i \\ 8i & -2+6i & -4i & 0 \\ 8-16i & 6-14i & -6+6i & -8i \\ -2-2i & -1 & 1+2i & -2+2i \end{bmatrix}$$

Matlab $\to$ basis for null(B^2):

$$\begin{aligned} \mathbf{u}_1 &= [2, 0, 4, -1+i]^T = \mathbf{v}_1 \\ \mathbf{u}_2 &= [-3-i, 4, 0, -2+2i]^T \end{aligned}$$

$\Rightarrow$ $\mathbf{u}_2$ is generalized eigenvector that is not an eigenvector.

Pick $\mathbf{v}_3 = \mathbf{u}_2 = [-3-i, 4, 0, -2+2i]^T$

Need: $B\mathbf{v}_3 = [-2, 0, -4, 1-i]^T = -\mathbf{v}_2$

Complex solution associated with $\mathbf{v}_3$:

$$\begin{aligned} \mathbf{z}_2(t) &= e^{(-1+i)t}(I\mathbf{v}_3 + tB\mathbf{v}_3) \\ &= e^{(-1+i)t}(\mathbf{v}_3 - t\mathbf{v}_2) \\ &= e^{(-1+i)t}[-3-i-2t, 4, -4t, \\ &\qquad -2+2i+(1-i)t]^T \end{aligned}$$

---

3. Take real and imaginary parts of $\mathbf{z}_1(t)$ and $\mathbf{z}_2(t)$ to obtain F.S.S:

$$\mathbf{x}_1(t) = \mathrm{Re}\,\mathbf{z}_1(t) = e^{-t}[2\cos t, 0, 4\cos t, -\cos t - \sin t]^T$$
$$\mathbf{x}_2(t) = \mathrm{Im}\,\mathbf{z}_1(t) = e^{-t}[2\sin t, 0, 4\sin t, \cos t - \sin t]^T$$
$$\mathbf{x}_3(t) = \mathrm{Re}\,\mathbf{z}_2(t) = e^{-t}[\sin t - (3+2t)\cos t, 4\cos t, -4t\cos t, (t-2)(\cos t + \sin t)]^T$$
$$\mathbf{x}_4(t) = \mathrm{Im}\,\mathbf{z}_2(t) = e^{-t}[-\cos t - (3+2t)\sin t, -4\sin t, -4t\sin t, (t-2)(\sin t - \cos t)]^T$$

Ex.: $A = \begin{bmatrix} 7 & 5 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 12 & 10 & -5 & 4 \\ -4 & -4 & 2 & -1 \end{bmatrix}$

Matlab $\rightarrow p(\lambda) = (\lambda+1)(\lambda-1)^3$

$\Rightarrow$ eigenvalues $\lambda_1 = -1$, $m_1 = 1$
$\lambda_2 = 1$, $m_2 = 3$

Find eigenvector for λ_1:

$$A + I = \begin{bmatrix} 8 & 5 & -3 & 2 \\ 0 & 2 & 0 & 0 \\ 12 & 10 & -4 & 4 \\ -4 & -4 & 2 & 0 \end{bmatrix}$$

Matlab $\rightarrow$ eigenvector (basis vector for null$(A+I)$): $\mathbf{v}_1 = [1, 0, 2, -1]^T$
Associated eigenvector solution:

$$\mathbf{x}_1(t) = e^{-t}[1, 0, 2, -1]^T$$

For $\lambda_2 = 1 \rightarrow$ check powers of $A - I$:

$$B \equiv A - I = \begin{bmatrix} 6 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 12 & 10 & -6 & 4 \\ -4 & -4 & 2 & -2 \end{bmatrix}$$

Matlab $\rightarrow$ basis of null(B):

$$\begin{aligned} \mathbf{v}_2 &= [1, 0, 2, 0]^T \\ \mathbf{v}_3 &= [1, -2, 0, 2]^T \end{aligned}$$

Associated eigenvector solutions:

$$\begin{aligned} \mathbf{x}_2(t) &= e^t[1, 0, 2, 0]^T \\ \mathbf{x}_3(t) &= e^t[1, -2, 0, 2]^T \end{aligned}$$

To find 4th solution check B^2:

$$B^2 = \begin{bmatrix} -8 & -8 & 4 & -4 \\ 0 & 0 & 0 & 0 \\ -16 & -16 & 8 & -8 \\ 8 & 8 & -4 & 4 \end{bmatrix}$$

$\Rightarrow RREF(B^2)$ has only one nonzero row $[1, 1, -1/2, 1/2]$.
Construct basis of null(B^2) by setting
$x_2, x_3 = 0, x_4 = 2 \rightarrow \mathbf{u}_1 = [-1, 0, 0, 2]^T$
$x_2, x_4 = 0, x_3 = 2 \rightarrow \mathbf{u}_2 = [1, 0, 2, 0]^T$
$x_3, x_4 = 0, x_2 = 1 \rightarrow \mathbf{u}_3 = [1, -1, 0, 0]^T$
Check which are *not* eigenvectors:

$$B\mathbf{u}_1 = -2\mathbf{v}_2,\ B\mathbf{u}_2 = \mathbf{0},\ B\mathbf{u}_3 = \mathbf{v}_2$$

$\Rightarrow$ Can choose $\mathbf{v}_4 = \mathbf{u}_1$ (simple).
Associated solution:

$$\begin{aligned} \mathbf{x}_4(t) &= e^t(I\mathbf{v}_4 + tB\mathbf{v}_4) = e^t(\mathbf{u}_1 - 2t\mathbf{v}_2) \\ &= e^t[-1-2t, 0, -4t, 2]^T \end{aligned}$$

Advanced Theory: Chains of Generalized Eigenvectors

Thm.: Let λ be an eigenvalue of a $n \times n$-matrix A with

- algebraic multiplicity m
- geometric multiplicity m_g

Let $B = A - \lambda I$ and k be s.t.

$$\begin{aligned} \dim \operatorname{null}(B^k) &= m \\ \dim \operatorname{null}(B^{k-1}) &< m \end{aligned}$$

There are m_g chains of vectors

$$\mathbf{v}_1^{(i)}, \ldots, \mathbf{v}_{r_i}^{(i)}, \quad 1 \le i \le m_g$$

s.t. $r_1 + r_2 + \cdots + r_{m_g} = m$,

$$\begin{aligned} B\mathbf{v}_{j+1}^{(i)} &= \mathbf{v}_j^{(i)}, \ 1 \le j < r_i \\ B\mathbf{v}_1^{(i)} &= \mathbf{0} \end{aligned}$$

and all the vectors $\mathbf{v}_j^{(i)}$ are a basis of $\operatorname{null}(B^k)$.

Computation of chains:

Assume $i-1$ chains have been computed. Let q be the largest integer for which there is a vector $\mathbf{v}$ in $\operatorname{null}(B^q)$ s.t. $B^{q-1}\mathbf{v} \neq \mathbf{0}$, and $\mathbf{v}$ and all previously computed chain-vectors are linearly independent. Set $r_i = q$. Then the ith chain is computed as

$$\begin{aligned} \mathbf{v}_{r_i}^{(i)} &= \mathbf{v} \\ \mathbf{v}_j^{(i)} &= B\mathbf{v}_{j+1}^{(i)} \text{ for } r_i > j \ge 1 \end{aligned}$$

Note: $B\mathbf{v}_1^i = \mathbf{0} \Rightarrow \mathbf{v}_1^i$ is eigenvector.

Solutions of $\mathbf{x}' = A\mathbf{x}$:

$$\begin{aligned} \mathbf{x}_j^{(i)}(t) &= e^{\lambda t}[\mathbf{v}_j^{(i)} + \sum_{l=1}^{j-1}(t^l/l!)\mathbf{v}_l^{(i)}] \text{ if } j > 1 \\ \mathbf{x}_1^{(i)}(t) &= e^{\lambda t}\mathbf{v}_1^{(i)} \end{aligned}$$

Single chain:

If $k = m \Rightarrow$ only one chain

$\mathbf{v}_1, \ldots, \mathbf{v}_m, \ \mathbf{v}_j = B\mathbf{v}_{j+1} \ (j < m)$

Ex.: $A = \begin{bmatrix} 3 & -3 & -6 & 5 \\ -3 & 2 & 5 & -4 \\ 2 & -6 & -4 & 7 \\ -3 & 0 & 5 & -2 \end{bmatrix}$

Matlab $\rightarrow p(\lambda) = (\lambda+1)^3(\lambda-2)$

$\Rightarrow$ eigenvalues $\lambda_1 = -1$, $m_1 = 3$

$\lambda_4 = 2, \quad m_4 = 1$

Eigenvector for λ_4: $\mathbf{v}_4 = [-1,1,1,2]^T$

$$\rightarrow \mathbf{x}_4(t) = e^{2t}[-1,1,1,2]^T$$

$\lambda_1 = -1$: Set $B = A + I$. Matlab $\rightarrow$ dim null$(B^2) = 2 \Rightarrow k = 3 \Rightarrow 1$ chain

Matlab's *null* $\rightarrow$ basis for null(B^3):

$$\begin{aligned} \mathbf{u}_1 &= [1,0,0,0]^T \\ \mathbf{u}_2 &= [0,0,1,0]^T \\ \mathbf{u}_3 &= [0,1,0,1]^T \end{aligned}$$

Find $B^2\mathbf{u}_1 \neq \mathbf{0} \Rightarrow$ chain can be generated by $\mathbf{u}_1$ (simple form). Set

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_1 = [1,0,0,0]^T \\ \mathbf{v}_2 &= B\mathbf{v}_3 = [4,-3,2,-3]^T \\ \mathbf{v}_1 &= B\mathbf{v}_2 = [-2,1,-2,1]^T \end{aligned}$$

Solutions associated with 3-chain:

$$\begin{aligned} \mathbf{x}_1(t) &= e^{-t}\mathbf{v}_1 = e^{-t}[-2,1,-2,1]^T \\ \mathbf{x}_2(t) &= e^{-t}(\mathbf{v}_2 + t\mathbf{v}_1) \\ &= e^{-t}[4+t,-3,2,-3]^T \\ \mathbf{x}_3(t) &= e^{-t}(\mathbf{v}_3 + t\mathbf{v}_2 + t^2\mathbf{v}_1/2) \\ &= e^{-t}[1+4t-t^2, -3t+t^2/2, \\ &\qquad 2t-t^2, -3t+t^2/2]^T \end{aligned}$$

Ex.: In example on p.8: $m_g = 2$, $k = 2$, $m = 3 \Rightarrow 2$ chains:

- $\mathbf{u}_1$, $-2\mathbf{v}_2$ is 2-chain
- $\mathbf{v}_3$ is 1-chain

Note: Approach via chains is "useful" if $m >> m_g$, especially if $m_g = 1$ and $k = m >> 1$.

Summary:

1. For any matrix A, a F.S.S $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ for $\mathbf{x}' = A\mathbf{x}$ can be computed using eigenvalues and (generalized) eigenvectors.

2. $X(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)]$ is a F.M: $\mathbf{x}(t) = X(t)\mathbf{c}$ is gen. sol.

3. M.E.: $e^{At} = X(t)(X(0))^{-1}$

$\mathbf{x}(t) = e^{At}\mathbf{x}(0)$

Matlab Tools

Eigenvalues & Eigenvectors

- *eig(A)*: vector of eigenvalues of A
- *[V,D]=eig(A)* $\rightarrow$ outputs
  V: matrix of eigenvectors (columns)
  D: diagonal matrix of eigenvalues

Symbolic Computation:

```
>> A=[1 1;-1 1];[V,D]=eig(sym(A))
V =               D=
[  1,  1]         [ 1+i,   0]
[  i, -i]         [   0, 1-i]
```

```
>> A=sym([-2 1 -1;1 -3 0;3 -5 0]);
>> [V,D]=eig(A)
V =               D =
[  1, -2]         [ -2,  0,  0]
[  1, -1]         [  0, -2,  0]
[  1,  1]         [  0,  0, -1]
```

Numerical Computation:

```
>> A=[-2 1 -1;1 -3 0;3 -5 0];
>> [V,D]=eig(A)
V =
    0.5774    0.5774   -0.8165
    0.5774    0.5774   -0.4082
    0.5774    0.5774    0.4082
D =
   -2.0000         0         0
         0   -2.0000         0
         0         0   -1.0000
```

Note: no generalized eigenvectors

Matrix Exponential

- *expm(A)*: matrix exponential of A

Symbolic Computation:

```
>> A=[1 1;-1 1];syms t;expm(sym(A)*t)
ans =
[  exp(t)*cos(t),  exp(t)*sin(t)]
[ -exp(t)*sin(t),  exp(t)*cos(t)]
```

Note: t must be declared symbolically

```
>> A=sym([-2 1 -1;1 -3 0;3 -5 0]);syms t;
>> expm(A*t);ans(:,1)
ans =
[ 3*exp(-2*t)-2*exp(-t)+2*t*exp(-2*t)]
[     2*t*exp(-2*t)-exp(-t)+exp(-2*t)]
[     2*t*exp(-2*t)+exp(-t)-exp(-2*t)]
```

In this example only first column of e^{At} is displayed

Numerical Computation:

```
>> A=[-2 1 -1;1 -3 0;3 -5 0];t=2;expm(A*t)
ans =
   -0.1425    0.3582   -0.1974
   -0.0438    0.1425   -0.0804
    0.1903   -0.3439    0.1720
```

Use loop to compute solution array:

```
>> t=linspace(0,1,20);x0=[1;0;0];x=[ ];
>> for n=1:20;x=[x expm(A*t(n))*x0];end
```

First entry of solution can be plotted via

```
>> plot(t,x(1,:))
```

Worked Out Examples from Exercises

Ex. 9.2.31: Find general solution of $\mathbf{y}' = A\mathbf{y}$ for $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

$T = 4$, $D = 4 \Rightarrow T^2 = 4D \Rightarrow$ single eigenvalue $\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{eigenvector } \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{Pick } \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow (A - 2I)\mathbf{w} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{v}$$

$$\Rightarrow \text{F.S.S.:} \quad \begin{aligned} \mathbf{y}_1(t) &= e^{2t}\mathbf{v} = e^{2t}[1,1]^T \\ \mathbf{y}_2(t) &= e^{2t}(\mathbf{w} + t\mathbf{v}) = e^{2t}([1,0]^T + t[1,1]^T) = e^{2t}[1+t, t]^T \end{aligned}$$

General solution:

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) = Y(t)\mathbf{c} \text{ with F.M. } Y(t) = e^{2t} \begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix}$$

Ex. 9.2.37: Find solution of system of Ex. 31 with IC $\mathbf{y}(0) = [2, -1]^T$

1st method: Match $\mathbf{c}$ to IC: $Y(0)\mathbf{c} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \mathbf{y}(t) = e^{2t} \begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = e^{2t} \begin{bmatrix} 2+3t \\ 3t-1 \end{bmatrix}$$

2nd method: $\mathbf{y}(t) = e^{At}\mathbf{y}(0) = e^{2t}(I + (A - 2I)t)\mathbf{y}(0)$

$$= e^{2t} \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} 2+3t \\ 3t-1 \end{bmatrix}$$

Ex. 9.6.1: Compute e^A for $A = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$ using the exponential series

$$A^2 = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^m = 0 \text{ if } m > 1$$

$$\Rightarrow e^A = I + A = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$$

Ex. 9.6.6: Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Show that $e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$

using $\left\{ \begin{array}{rcl} \cos t & = & 1 - t^2/2! + t^4/4! - t^6/6! + \cdots \\ \sin t & = & t - t^3/3! + t^5/5! - t^7/7! + \cdots \end{array} \right\}$. Compute:

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -I,\ A^3 = A(-I) = -A,\ A^4 = -A^2 = -(-I) = I$$

$$\Rightarrow\ A^{4m} = I,\ A^{4m+1} = A,\ A^{4m+2} = -I,\ A^{4m+3} = -A \quad \text{for } m = 0, 1, 2, \ldots \ \Rightarrow$$

$$\begin{aligned} e^{At} &= \sum_{n=0}^{\infty} (At)^n/n! = \sum_{m=0}^{\infty} \left(I \frac{t^{4m}}{(4m)!} + A \frac{t^{4m+1}}{(4m+1)!} - I \frac{t^{4m+2}}{(4m+2)!} - A \frac{t^{4m+3}}{(4m+3)!}\right) \\ &= I(1 - t^2/2! + t^4/4! - t^6/6! + \cdots) + A(t - t^3/3! + t^5/5! - t^7/7! + \cdots) \\ &= I \cos t + A \sin t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \end{aligned}$$

Ex. 9.6.11: Compute e^{At} by diagonalizing A for $A = \begin{bmatrix} -2 & 6 \\ 0 & -1 \end{bmatrix}$

A upper triangular $\Rightarrow$ eigenvalues are diagonal entries: $\lambda_1 = -2$, $\lambda_2 = -1$

$A-(-2)I = \begin{bmatrix} 0 & 6 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_1 = [1,0]^T$. $A-(-1)I = \begin{bmatrix} -1 & 6 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = [6,1]^T$

Set $V = [\mathbf{v}_1, \mathbf{v}_1] = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \Rightarrow V^{-1} = \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}$. Verify $V^{-1}AV$ is diagonal:

$$V^{-1}AV = \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 12 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \equiv D$$

$$\begin{aligned} \Rightarrow A = VDV^{-1} \Rightarrow e^{At} &= Ve^{Dt}V^{-1} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} & 6e^{-t} \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} & 6e^{-t} - 6e^{-2t} \\ 0 & e^{-t} \end{bmatrix} \end{aligned}$$

Ex. 9.6.14: Compute e^{At} for $A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$

C.P.: $p(\lambda) = \left| \begin{matrix} -2-\lambda & 1 \\ -1 & -\lambda \end{matrix} \right| = (-2-\lambda)(-\lambda) + 1 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$

$\Rightarrow$ eigenvalue $\lambda = -1$. Set $B = A + I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Compute:

$B^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow e^{At} = e^{-t}(I+Bt) = e^{-t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$

Ex. 9.6.18: Compute e^{At} for $A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ -2 & 4 & -3 \end{bmatrix}$

$$\begin{aligned} \text{C.P.: } p(\lambda) &= \begin{vmatrix} -1-\lambda & 0 & 0 \\ -1 & 1-\lambda & -1 \\ -2 & 4 & -3-\lambda \end{vmatrix} = -(\lambda+1) \begin{vmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{vmatrix} \\ &= -(\lambda+1)[(\lambda-1)(\lambda+3)+4] = -(\lambda+1)(\lambda^2+2\lambda+1) \\ &= -(\lambda+1)^3 \end{aligned}$$

$\Rightarrow$ single eigenvalue $\lambda = -1$. Set $B = A + I$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -2 & 4 & -2 \end{bmatrix} \;\Rightarrow\; B^2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -2 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B^2 = 0 \;\; (k=2) \;\;\Rightarrow\;\; e^{At} = e^{-t}(I + Bt) = e^{-t} \begin{bmatrix} 1 & 0 & 0 \\ -t & 1+2t & -t \\ -2t & 4t & 1-2t \end{bmatrix}$$

Note: That $B^2 = 0$ follows also directly from $\dim \operatorname{null}(B) = 2$

Ex. 9.6.22: Compute e^{At} for $A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}$

$$\text{C.P.: } p(\lambda) = \begin{vmatrix} 1-\lambda & -1 & 2 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & -1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ -1 & 2 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)^4$$

$\Rightarrow$ single eigenvalue $\lambda = 1$. Set $B = A - I = \begin{bmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}$

$$\Rightarrow B^2 = \begin{bmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow e^{At} = e^t(I + Bt) = e^t \begin{bmatrix} 1 & -t & 2t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -t & 2t & 1 \end{bmatrix}$$

Ex. 9.6.26: Do the 6 tasks below for $A = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{bmatrix}$ by hand

1. Find eigenvalues:

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} -2-\lambda & 1 & -1 \\ 1 & -3-\lambda & 0 \\ 3 & -5 & -\lambda \end{vmatrix} \\ &= (-1)^{2+1}1\begin{vmatrix} 1 & -1 \\ -5 & -\lambda \end{vmatrix} + (-1)^{2+2}(-3-\lambda)\begin{vmatrix} -2-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} \\ &= -(-\lambda-5) - (\lambda+3)[(\lambda+2)\lambda+3] = \lambda+5-(\lambda+3)(\lambda^2+2\lambda+3) \\ &= \lambda+5-(\lambda^3+4\lambda^2+9\lambda+9) = -(\lambda^3+5\lambda^2+8\lambda+4) \\ &= -(\lambda+1)(\lambda+2)^2 \quad \Rightarrow \text{ eigenvalues } \lambda=-1 \text{ and } \lambda=-2 \end{aligned}$$

2. Find algebraic (m) and geometric (m_g) multiplicities for each eigenvalue:

$$\lambda=-1 \to m=1; \;\; \lambda=-2 \to m=2 \qquad \text{(from } p(\lambda))$$

Find geometric multiplicities:

$\lambda=-1$: $A+I = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & 0 \\ 3 & -5 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow m_g = 1$

$\lambda=-2$: $A+2I \equiv B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow m_g = 1$

Ex. 9.6.26 continued 1

3. For each eigenvalue find smallest k s.t. dim null$((A-\lambda I)^k)=m$

For $\lambda=-1$: $k=1$ (since $m=m_g=1$)

For $\lambda=-2$:

$$B^2=\begin{bmatrix}0 & 1 & -1\\ 1 & -1 & 0\\ 3 & -5 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & -1\\ 1 & -1 & 0\\ 3 & -5 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4 & -2\\ -1 & 2 & -1\\ 1 & -2 & 1\end{bmatrix}\to\begin{bmatrix}1 & -2 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\Rightarrow \dim\text{null}(B^2)=2=m$. Since $\dim\text{null}(B)=1\Rightarrow k=2$

4. For each eigenvalue, find m linearly independent generalized eigenvectors.

For $\lambda=-1$: From $RREF(A+I)\Rightarrow$ eigenvector $\mathbf{v}_1=[-2,-1,1]^T$

For $\lambda=-2$: From $RREF(B)\Rightarrow$ eigenvector $\mathbf{v}_2=[1,1,1]^T$; $(B=A+2I)$

$m=2\to$ need solution of $B^2\mathbf{v}=\mathbf{0}$ that is *not* a multiple of $\mathbf{v}_2$

Use $RREF(B^2)$: set $y_2=0$, $y_3=1\Rightarrow y_1=-1\Rightarrow \mathbf{v}_3\equiv[-1,0,1]^T$ is in null(B^2)

Since $\mathbf{v}_2,\mathbf{v}_3$ are linearly independent, they are linearly independent generalized eigenvectors for $\lambda=-2$.

5. Verify linear independence of all generalized eigenvectors from 4.

Set $V=[\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3]=\begin{bmatrix}-2 & 1 & -1\\ -1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}$. Compute determinant of V:

$$\det(V)=(-1)^{2+1}(-1)\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}+(-1)^{2+2}1\begin{vmatrix}-2 & -1\\ 1 & 1\end{vmatrix}=2-1=1\neq 0$$

$\Rightarrow \mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ are linearly independent.

Ex. 9.6.26 continued 2

6. Find fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$

$\lambda = -1$, $\mathbf{v}_1 \to$ eigenvector solution: $\mathbf{y}_1(t) = e^{-t}[-2, -1, 1]^T$

$\lambda = -2$, $\mathbf{v}_2 \to$ eigenvector solution: $\mathbf{y}_2(t) = e^{-2t}[1, 1, 1]^T$

$\lambda = -2$, $\mathbf{v}_3 \to$ generalized eigenvector solution $\mathbf{y}_3(t) = e^{-2t}(\mathbf{v}_3 + tB\mathbf{v}_3)$
Compute

$$B\mathbf{v}_3 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = -\mathbf{v}_2$$

$$\begin{aligned} \Rightarrow \mathbf{y}_3(t) &= e^{-2t}(\mathbf{v}_3 - t\mathbf{v}_2) \\ &= e^{-2t}([-1, 0, 1]^T - t[1, 1, 1]^T) \\ &= e^{-2t}[-1 - t, -t, 1 - t]^T \end{aligned}$$

and $\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)$ are F.S.S. for $\mathbf{y}' = A\mathbf{y}$.

Ex. 9.6.33: 6 tasks as in Ex. 26 for $A=\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 \\ -14 & -2 & -7 & 11 & -9 & -8 \\ -9 & -3 & -3 & 7 & -6 & -4 \\ -19 & -5 & -9 & 17 & -12 & -9 \\ -29 & -7 & -13 & 23 & -16 & -15 \\ 19 & 5 & 9 & -15 & 12 & 11 \end{bmatrix}$

1. Find eigenvalues:

```
>> A=[2 0 0 0 0 1;-14 -2 -7 11 -9 -8;-9 -3 -3 7 -6 -4;...
-19 -5 -9 17 -12 -9;-29 -7 -13 23 -16 -15;19 5 9 -15 12 11];
>> A=sym(A);factor(poly(A))
ans =
(x-1)^3*(x-2)^3
```

$\Rightarrow$ $\lambda = 1, 2$ both with $m = 3$

2. Find algebraic (m) and geometric (m_g) multiplicities:
From $p(\lambda)$: $m = 3$ for $\lambda = 1$ and $\lambda = 2$. Find m_g:

```
>> B1=A-eye(6);null(B1)'
ans =
[ -1,  0, -2, -1,  1,  1]
```

```
>> B2=A-2*eye(6);null(B2)'
ans =
[  1,  7, -6,  0,  0,  0]
[  0,  3, -3,  0,  1,  0]
[  0, -6,  5,  1,  0,  0]
```

$\Rightarrow$ $m_g = 1$ for $\lambda = 1$; $m_g = 3 = m$ for $\lambda = 2$

3. For each eigenvalue find smallest k s.t. dim null$((A - \lambda I)^k) = m$
For $\lambda = 2$ we are done: from 2. $\Rightarrow k = 1$. Check $\lambda = 1$:

```
>> null(B1^2)'
ans =
[  1, -2,  0, -1, -3,  1]
[  0,  1,  1,  1,  1, -1]
```

```
>> null(B1^3)'
ans =
[ -1,   2,  1,  0,  0,  0]
[ -1, 3/2,  0,  0,  1,  0]
[ -2, 5/2,  0, -1,  0,  1]
```

$\Rightarrow k = 3$

Ex. 9.6.33 continued 1

4. For each eigenvalue, find m linearly independent generalized eigenvectors.

(a) For $\lambda = 2$ we are done: $m_g = m = 3$
$\Rightarrow$ every generalized eigenvector is in null$(A - 2I)$ and so is an eigenvector.

Assign variables to the eigenvectors in Matlab and denote them by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

```
>> null(B2);v1=ans(:,1);v2=ans(:,2);v3=ans(:,3);
>> [v1 v2 v3]'
ans =
[  0,   3, -3,   0,   1,   0]
[  1,   7, -6,   0,   0,   0]
[  0, -6,   5,   1,   0,   0]
```

$$\begin{aligned} \mathbf{v}_1 &= [0, 3, -3, 0, 1, 0]^T \\ \mathbf{v}_2 &= [1, 7, -6, 0, 0, 0]^T \\ \mathbf{v}_3 &= [0, -6, 5, 1, 0, 0]^T \end{aligned}$$

(b) For $\lambda = 1$ we need a basis of null$((A - I)^3)$. In view of task 6 it makes sense to determine a chain of 3 generalized eigenvectors. Let's check the chains for each of the basis vectors:

```
>> null(B1^3);u1=ans(:,1);u2=ans(:,2);u3=ans(:,3);
>> [u1 B1*u1 B1^2*u1]',[u2 B1*u2 B1^2*u2]',[u3 B1*u3 B1^2*u3]'

ans =                | ans =                           | ans =
[-1,2, 1, 0,0,0]     | [  -1,3/2,    0,    0,   1,   0] | [  -2,5/2,    0,   -1,   0,    1]
[-1,1,-1, 0,2,0]     | [  -1,1/2,-3/2,-1/2,3/2,1/2]     | [  -1,3/2,-1/2, 1/2,5/2,-1/2]
[-1,0,-2,-1,1,1]     | [-1/2,  0,   -1,-1/2,1/2,1/2]    | [-3/2,  0,   -3,-3/2,3/2, 3/2]
```

All three basis vectors generate full chains (this needs not to be the case in general). Let's choose the simplest chain which is the first.

Assign names to chain vectors in Matlab; denote them $\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4$:

Ex. 9.6.33 continued 2

```
>> v6=u1;v5=B1*v6;v4=B1*v5;
>> [v6 v5 v4]'
ans =
[ -1,  2,  1,  0,  0,  0]
[ -1,  1, -1,  0,  2,  0]
[ -1,  0, -2, -1,  1,  1]
```

$$\begin{aligned}
\mathbf{v}_6 &= [-1,2,1,0,0,0]^T \\
\mathbf{v}_5 &= (A-I)\mathbf{v}_6 = [-1,1,-1,0,2,0]^T \\
\mathbf{v}_4 &= (A-I)\mathbf{v}_5 = [-1,0,-2,-1,1,1]^T \\
\mathbf{0} &= (A-I)\mathbf{v}_4 \quad (\mathbf{v}_4 \text{ is eigenvector})
\end{aligned}$$

5. Verify linear independence of all generalized eigenvectors from 4.

```
>> det([v1 v2 v3 v4 v5 v6])
ans =
-1
```

Since $\det([\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\mathbf{v}_5,\mathbf{v}_6]) = -1 \neq 0$, the six generalized eigenvectors are linearly independent.

6. Grand Finale: Find fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent eigenvectors for $\lambda = 2$
$\Rightarrow$ three linearly independent eigenvector solutions:

$$\begin{aligned}
\mathbf{y}_1(t) &= e^{2t}\mathbf{v}_1 = e^{2t}[0,3,-3,0,1,0]^T \\
\mathbf{y}_2(t) &= e^{2t}\mathbf{v}_2 = e^{2t}[1,7,-6,0,0,0]^T \\
\mathbf{y}_3(t) &= e^{2t}\mathbf{v}_3 = e^{2t}[0,-6,5,1,0,0]^T
\end{aligned}$$

Solutions associated with chain $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$:

$$\begin{aligned}
\mathbf{y}_4(t) &= e^t\mathbf{v}_4 = e^t[-1,0,-2,-1,1,1]^T \\
\mathbf{y}_5(t) &= e^t(\mathbf{v}_5 + t\mathbf{v}_4) = e^t[-1-t,1,-1-2t,-t,2+t,t]^T \\
\mathbf{y}_6(t) &= e^t(\mathbf{v}_6 + t\mathbf{v}_5 + t^2\mathbf{v}_4/2) \\
&= e^t[-1-t-t^2/2, 2+t, 1-t-t^2, -t^2/2, 2t+t^2/2, t^2/2]^T
\end{aligned}$$

Note: Ex. 35-45 require same tasks as Ex. 26-33, except task 5.