9.5: Fundamental Sets of Eigenvector Solutions

Homogenous system:

$$\mathbf{x}' = A\mathbf{x}, \quad A : n \times n$$

Characteristic Polynomial: (degree n)

$$p(\lambda) = \det(A - \lambda I)$$

Def.: The multiplicity of a root λ_i of $p(\lambda)$ is the smallest positive integer m for which $\frac{d^m}{d\lambda^m} p(\lambda)|_{\lambda=\lambda_i} \neq 0$.

Fundamental Thm. of Algebra: If the roots are counted with multiplicities, then $p(\lambda)$ has exactly n roots $\lambda_1, \ldots, \lambda_n$, and

$$p(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

Thm.: If $\lambda_1, \ldots, \lambda_n$ are n real eigenvalues of A and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent eigenvectors, then

$$\mathbf{x}_j(t) = e^{\lambda_j t} \mathbf{v}_j, \ 1 \leq j \leq n$$

are a fundamental set of solutions.

Ex.: $A = \begin{bmatrix} 8 & -5 & 10 \\ 2 & 1 & 2 \\ -4 & 4 & -6 \end{bmatrix}$. Find

$$\begin{aligned} p(\lambda) &= -\lambda^3 + 3\lambda^2 + 4\lambda - 12 \\ &= -(\lambda + 2)(\lambda - 3)(\lambda - 2) \end{aligned}$$

$$\Rightarrow \lambda_1 = -2, \ \lambda_2 = 3, \ \lambda_3 = 2.$$

Eigenvectors:

$$A + 2I = \begin{bmatrix} 10 & -5 & 10 \\ 2 & 3 & 2 \\ -4 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \mathbf{v}_1 = [-1, 0, 1]^T$. Analogously: $\mathbf{v}_2 = [1, 1, 0]^T$, $\mathbf{v}_3 = [0, 2, 1]^T$.

$$\Rightarrow \begin{aligned} \mathbf{x}_1(t) &= e^{-2t}[-1, 0, 1]^T \\ \mathbf{x}_2(t) &= e^{3t}[1, 1, 0]^T \\ \mathbf{x}_3(t) &= e^{2t}[0, 2, 1]^T \end{aligned}$$

are a fundamental set of solutions.

Ex.: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$p(\lambda) = (1 - \lambda)^2 + 1 = 0 \Rightarrow \lambda_{1,2} = 1 \pm i$

$\Rightarrow$**Complex Eigenvalues**

Review: Complex Numbers and Complex Exponential

Complex Numbers

$\mathbf{C}$: set of complex numbers

$$\lambda = \alpha + i\beta \in \mathbf{C}, \quad i = \sqrt{-1}$$

$\alpha, \beta \in \mathbf{R}$; $\alpha = \text{Re}(\lambda)$, $\beta = \text{Im}(\lambda)$

Complex conjugate: $\bar{\lambda} = \alpha - i\beta$

Addition and multiplication

If $\lambda = \alpha + i\beta$, $\mu = \gamma + i\delta$:

$$\begin{aligned} \lambda + \mu &= (\alpha+\gamma) + i(\beta+\delta) \\ &\quad \text{like vector addition} \\ \lambda\mu &= (\alpha + i\beta)(\gamma + i\delta) \\ &= \alpha\gamma - \beta\delta + i(\alpha\delta + \beta\gamma) \\ \text{Re}(\lambda) &= (\lambda + \bar{\lambda})/2 \\ \text{Im}(\lambda) &= (\lambda - \bar{\lambda})/(2i) \end{aligned}$$

Absolute value: $|\lambda| = \sqrt{\alpha^2 + \beta^2}$

Inversion: $\frac{1}{\lambda} = \frac{\bar{\lambda}}{\lambda\bar{\lambda}} = \frac{\bar{\lambda}}{|\lambda|^2}$

Euler Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\rightarrow$ point on the unit circle:

$$|e^{i\theta}|^2 = \cos^2\theta + \sin^2\theta = 1$$

$$e^{-i\theta} = \cos\theta - i\sin\theta = 1/e^{i\theta}$$

Complex exponential:

$$\begin{aligned} e^{\lambda} &= e^{\alpha + i\beta} = e^{\alpha}e^{i\beta} \\ &= e^{\alpha}(\cos\beta + i\sin\beta) \\ e^{\bar{\lambda}} &= e^{\alpha - i\beta} = e^{\alpha}e^{-i\beta} \\ &= e^{\alpha}(\cos\beta - i\sin\beta) = \overline{(e^{\lambda})} \end{aligned}$$

Exponential function:

$$\begin{aligned} e^{\lambda t} &= e^{\alpha t}\Big(\cos(\beta t) + i\sin(\beta t)\Big) \\ e^{\bar{\lambda} t} &= e^{\alpha t}\Big(\cos(\beta t) - i\sin(\beta t)\Big) \\ &= \overline{(e^{\lambda t})} \end{aligned}$$

Complex Eigenvalues and Eigenvectors

Complex vectors:

$$\mathbf{x}, \mathbf{y} \in \mathbf{R}^n \to \mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbf{C}^n$$

Complex matrices:

matrices with complex entries

nullspace, linear independence, basis, dimension as in the real case (with complex vectors and scalars allowed)

Def.: Let A: real $n \times n$-matrix.

- A complex root of
$$\det(A - \lambda I) = 0$$
is a complex eigenvalue.
- Any $\mathbf{0} \neq \mathbf{v} \in \mathbf{C}^n$ s.t.
$$A\mathbf{v} = \lambda \mathbf{v}$$
is an eigenvector, and
- null$(A - \lambda I)$ is the eigenspace for λ.

Pairs: $A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$

$\Rightarrow$ complex conjugate pairs of eigenvalues and eigenvectors

Thm.: Let λ be a complex eigenvalue with eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$. Then $\mathbf{v}, \overline{\mathbf{v}}$ and $\mathbf{u}, \mathbf{w}$ are linearly independent.

Ex.: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \rightarrow \left\{ \begin{matrix} \lambda_1 & = & 1+i \\ \lambda_2 & = & 1-i \end{matrix} \right\}$

Compute $A - (1+i)I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$

$$\overset{R3(1,i)}{\longrightarrow} \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \overset{R1(2,1,1)}{\longrightarrow} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \mathbf{v} = [1, i]^T = [1, 0]^T + i[0, 1]^T$
$= \mathbf{u} + i\mathbf{w}$ is basis of null$(A - \lambda_1 I)$

$\Rightarrow \mathbf{v}$ is eigenvector for $\lambda_1 = 1 + i$

$\Rightarrow \overline{\mathbf{v}} = [1, -i]^T = \mathbf{u} - i\mathbf{w}$ is eigenvector for $\lambda_2 = \overline{\lambda_1} = 1 - i$

Since $\left\{ \begin{matrix} \det([\mathbf{v}, \overline{\mathbf{v}}]) & = & -2i & \neq & 0 \\ \det([\mathbf{u}, \mathbf{w}]) & = & 1 & \neq & 0 \end{matrix} \right\}$,

$\mathbf{v}, \overline{\mathbf{v}}$ and $\mathbf{u}, \mathbf{w}$ are linearly independent.

Solutions of Systems Derived from Complex Conjugate Pairs of Eigenvalues and Eigenvectors

A: real $n \times n$-matrix

System: $\quad \mathbf{x}' = A\mathbf{x} \qquad (1)$

Assume $A\mathbf{v} = \lambda\mathbf{v}$ with

$$\begin{aligned} \lambda &= \alpha + i\beta \in \mathbf{C}, \quad \beta \neq 0 \\ \mathbf{v} &= \mathbf{u} + i\mathbf{w} \in \mathbf{C}^n, \quad \mathbf{v} \neq \mathbf{0} \end{aligned}$$

Linearly independent

complex solutions of (1):

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{v}, \quad \overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{v}}$$

Real and imaginary parts:

$$\begin{aligned} \mathbf{z}(t) &= e^{(\alpha+i\beta)t}(\mathbf{u} + i\mathbf{w}) \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{w}) \\ &= e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t) \\ &\quad + i e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t) \\ &= \operatorname{Re}\mathbf{z}(t) + i\operatorname{Im}\mathbf{z}(t) \end{aligned}$$

Linearly independent

real solutions of (1):

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t) \\ \mathbf{x}_2(t) &= e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t) \end{aligned}$$

For 2d Systems:

($n = 2$, Sec. 9.2)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \left\{ \begin{array}{lcl} T &=& a + d \\ D &=& ad - bc \end{array} \right\}$$

$$p(\lambda) = \lambda^2 - T\lambda + D$$

Assume $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues $\lambda, \overline{\lambda}$:

$$\begin{aligned} \lambda &= (T + i\sqrt{4D - T^2})/2 \\ &\equiv \alpha + i\beta \end{aligned}$$

Let $\mathbf{v} = \mathbf{u} + i\mathbf{w} \neq \mathbf{0}$ be in null$(A - \lambda I)$. Then

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t) \\ \mathbf{x}_2(t) &= e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t) \end{aligned}$$

are already a **fundamental set of solutions**

Ex.: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow T = 2,\ D = 2$

$T^2 - 4D = -4 < 0 \Rightarrow$ complex eigenvalues

Characteristic polynomial:

$$\begin{aligned} p(\lambda) &= \lambda^2 - 2\lambda + 2 \\ &= (\lambda - 1)^2 + 1 \\ \Rightarrow (\lambda - 1)^2 &= -1 \\ \Rightarrow \lambda - 1 &= \pm i \end{aligned}$$

Thus the eigenvalues are:

$\lambda = 1 + i,\ \ \overline{\lambda} = 1 - i \quad (\alpha = 1,\ \beta = 1)$

Find eigenvector for $\lambda = 1 + i$:

$$A - (1 + i)I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

$A - \lambda I$ is singular

- $\Rightarrow$ rows of $A - \lambda I$ are linearly dependent
- $\Rightarrow$ 2nd row is complex multiple of 1st row ($-i[-i, 1] = [-1, -i]$)
- $\Rightarrow$ need only consider 1st row for finding basis $\mathbf{v} = [v_1, v_2]^T$ of null$(A - \lambda I)$

Equation from 1st row: $-iv_1 + v_2 = 0$

Set $v_2 = i \Rightarrow v_1 = 1 \Rightarrow$ eigenvector:

$$\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Lin. indep. complex solutions:

$$\mathbf{z}(t) = e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix},\ \overline{\mathbf{z}}(t) = e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Fundamental set of real solutions:

$$\begin{aligned} \mathbf{x}_1(t) &= e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) \\ &= e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \\ \mathbf{x}_2(t) &= e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t \right) \\ &= e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \end{aligned}$$

Fundamental matrix:

$$X(t) = e^t \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Fundamental Set of Eigenvector Solutions

A: real $n \times n$-matrix

System: $\mathbf{x}' = A\mathbf{x}$ (1)

Thm.: Assume

1. A has k real eigenvalues $\lambda_1, \dots, \lambda_k$ with linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$.
2. A has m complex conjugate pairs $\lambda_{k+1}, \overline{\lambda}_{k+1}, \dots, \lambda_{k+m}, \overline{\lambda}_{k+m}$ of eigenvalues with eigenvectors $\mathbf{v}_{k+1}, \overline{\mathbf{v}}_{k+1}, \dots, \mathbf{v}_{k+m}, \overline{\mathbf{v}}_{k+m}$.
3. $n = k + 2m$ and the eigenvectors of 2. are linearly independent.

Then the n vector functions

$$\begin{aligned}
\mathbf{x}_i(t) &= e^{\lambda_i t}\mathbf{v}_i, \quad 1 \le i \le k \\
\mathbf{x}_j(t) &= e^{\alpha_j t}(\mathbf{u}_j \cos \beta_j t - \mathbf{w}_j \sin \beta_j t) \\
\mathbf{x}_{j+m}(t) &= e^{\alpha_j t}(\mathbf{u}_j \sin \beta_j t + \mathbf{w}_j \cos \beta_j t) \\
& \quad k + 1 \le j \le k + m
\end{aligned}$$

are a fundamental set of solutions.

Ex.: $A = \begin{bmatrix} 14 & 8 & -19 \\ -40 & -25 & 52 \\ -5 & -4 & 6 \end{bmatrix}$

Use Matlab's *poly* and *factor*

$\Rightarrow p(\lambda) = -(\lambda + 1)[(\lambda + 2)^2 + 9]$

$\Rightarrow$ eigenvalues: $\lambda_1 = -1$
$\lambda_2 = -2 + 3i$, $\lambda_3 = \overline{\lambda_2}$

eigenvectors (using Matlab's *null*):

$$\begin{aligned}
\mathbf{v}_1 &= [2, 1, 2]^T \\
\mathbf{v}_2 &= [i, 2 - 2i, 1]^T \\
&= [0, 2, 1]^T + i[1, -2, 0]^T
\end{aligned}$$

Fundamental set of solutions:

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{-t}[2, 1, 2]^T \\
\mathbf{x}_2(t) &= e^{-2t}([0, 2, 1]^T \cos 3t \\
& \quad -[1, -2, 0]^T \sin 3t) = e^{-2t}\mathbf{p}(t) \\
\mathbf{x}_3(t) &= e^{-2t}([0, 2, 1]^T \sin 3t \\
& \quad +[1, -2, 0]^T \cos 3t) = e^{-2t}\mathbf{q}(t)
\end{aligned}$$

$\mathbf{p}(t) = [-\sin 3t, 2\cos 3t + 2\sin 3t, \cos 3t]^T$

$\mathbf{q}(t) = [\cos 3t, 2\sin 3t - 2\cos 3t, \sin 3t]^T$

Worked Out Examples from Exercises

Ex. 9.2.18: Find fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$ for $A = \begin{bmatrix} -1 & 1 \\ -5 & -5 \end{bmatrix}$

$$T = -6,\ D = 10 \Rightarrow T^2 - 4D = -4 < 0 \Rightarrow \text{complex eigenvalues:}$$

$$p(\lambda) = \lambda^2 + 6\lambda + 10 = (\lambda+3)^2 + 1 = 0 \ \Rightarrow\ \text{eigenvalues } \lambda = -3+i,\ \overline{\lambda} = -3-i$$

$$A-(-3+i)I = \begin{bmatrix} 2-i & 1 \\ -5 & -2-i \end{bmatrix} \Rightarrow \text{eigenvector: } \mathbf{v} = \begin{bmatrix} 2+i \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Rightarrow \mathbf{u} = [2, -5]^T$, $\mathbf{v} = [1, 0]^T \Rightarrow$ fundamental set of solutions:

$$\begin{aligned} \mathbf{y}_1(t) &= e^{-3t}\Big(\begin{bmatrix} 2 \\ -5 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \Big) = e^{-3t} \begin{bmatrix} 2\cos t - \sin t \\ -5\cos t \end{bmatrix} \\ \mathbf{y}_2(t) &= e^{-3t}\Big(\begin{bmatrix} 2 \\ -5 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t \Big) = e^{-3t} \begin{bmatrix} 2\sin t + \cos t \\ -5\sin t \end{bmatrix} \end{aligned}$$

Ex. 9.2.24: Find solution of system of Ex. 18 for IC $\mathbf{y}(0) = [1, -5]^T$

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \mathbf{y}(t) = e^{-3t}\Big(\begin{bmatrix} 2\cos t - \sin t \\ -5\cos t \end{bmatrix} - \begin{bmatrix} 2\sin t + \cos t \\ -5\sin t \end{bmatrix} \Big) = e^{-3t} \begin{bmatrix} \cos t - 3\sin t \\ 5\sin t - 5\cos t \end{bmatrix}$$

Ex. 9.5.3: Find eigenvalues and eigenvectors of A and verify linear independence of eigenvectors

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -6 & 1 & -4 \\ -3 & 0 & -1 \end{bmatrix}, \ \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -6 & 1-\lambda & -4 \\ -3 & 0 & -1-\lambda \end{vmatrix}$$

$$\Rightarrow \ p(\lambda) = (2-\lambda)\begin{vmatrix} 1-\lambda & -4 \\ 0 & -1-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-1)(\lambda+1)$$

$\Rightarrow$ eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$. Find eigenvectors:

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -6 & -1 & -4 \\ -3 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \ \Rightarrow \ \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 0 & -4 \\ -3 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \ \Rightarrow \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A + I = \begin{bmatrix} 3 & 0 & 0 \\ -6 & 2 & -4 \\ -3 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \ \Rightarrow \ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\det([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \begin{vmatrix} -1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$$

$\Rightarrow \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent

Ex. 9.5.9: Find general solution of $\begin{cases} x' &= -3x \\ y' &= -5x+6y-4z \\ z' &= -5x+2y \end{cases}$

$$A = \begin{bmatrix} -3 & 0 & 0 \\ -5 & 6 & -4 \\ -5 & 2 & 0 \end{bmatrix}, \ \det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 0 & 0 \\ -5 & 6-\lambda & -4 \\ -5 & 2 & -\lambda \end{vmatrix}$$

$$\Rightarrow p(\lambda) = (-3-\lambda)\begin{vmatrix} 6-\lambda & -4 \\ 2 & -\lambda \end{vmatrix} = -(\lambda+3)[(6-\lambda)(-\lambda)+8]$$

$$= -(\lambda+3)(\lambda^2-6\lambda+8) = -(\lambda+3)(\lambda-2)(\lambda-4)$$

$$\lambda_1 = -3 \to A+3I = \begin{bmatrix} 0 & 0 & 0 \\ -5 & 9 & -4 \\ -5 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 5 & -2 & -3 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 5 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \mathbf{v}_1 = [1,1,1]^T$

$$\lambda_2 = 2 \to A-2I = \begin{bmatrix} -5 & 0 & 0 \\ -5 & 4 & -4 \\ -5 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = [0,1,1]^T$$

$$\lambda_3 = 4 \to A-4I = \begin{bmatrix} -7 & 0 & 0 \\ -5 & 2 & -4 \\ -5 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_3 = [0,2,1]^T$$

General Solution: $\mathbf{x}(t) = c_1 e^{-3t}[1,1,1]^T + c_2 e^{2t}[0,1,1]^T + c_3 e^{4t}[0,2,1]^T$

Ex. 9.5.15: Find solution to IC $\mathbf{x}(0) = [-2, 0, 2]^T$ for system of Ex. 9
Write general solution as $\mathbf{x}(t) = X(t)\mathbf{c}$, $\mathbf{c} = [c_1, c_2, c_3]^T$, with F.M.

$$X(t) = \begin{bmatrix} e^{-3t} & 0 & 0 \\ e^{-3t} & e^{2t} & 2e^{4t} \\ e^{-3t} & e^{2t} & e^{4t} \end{bmatrix};\ \text{IC} \Rightarrow X(0)\mathbf{c} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$[X(0), \mathbf{x}(0)] = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -2 \end{bmatrix} \Rightarrow \mathbf{c} = \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow\ \mathbf{x}(t) &= -2e^{-3t}[1,1,1]^T + 6e^{2t}[0,1,1]^T - 2e^{4t}[0,2,1]^T \\ &= [-2e^{-3t}, -2e^{-3t} + 6e^{2t} - 4e^{4t}, -2e^{-3t} + 6e^{2t} - 2e^{4t}]^T \end{aligned}$$

Ex. 9.5.19: Find real and imaginary parts for $\mathbf{y}(t) = e^{2it}[1, 1+2i, -3i]^T$
Write $\mathbf{y}(t) = e^{2it}\mathbf{v}$, $\mathbf{v} = [1, 1+2i, -3i]^T = \mathbf{u} + i\mathbf{w}$, $\mathbf{u} = [1,1,0]^T$, $\mathbf{w} = [0,2,-3]^T$

$$\begin{aligned} \mathbf{y}(t) &= (\cos 2t + i \sin 2t)(\mathbf{u} + i\mathbf{w}) = \mathbf{u}\cos 2t + i\mathbf{u}\sin 2t + i\mathbf{w}\cos 2t + i^2\mathbf{w}\sin 2t \\ &= (\mathbf{u}\cos 2t - \mathbf{w}\sin 2t) + i(\mathbf{u}\sin 2t + \mathbf{w}\cos 2t) \end{aligned}$$

$$\begin{aligned} \Rightarrow\ \operatorname{Re}\mathbf{y}(t) &= \mathbf{u}\cos 2t - \mathbf{w}\sin 2t = [1,1,0]^T\cos 2t - [0,2,-3]^T \sin 2t \\ &= [\cos 2t, \cos 2t - 2\sin 2t, 3\sin 2t]^T \\ \operatorname{Im}\mathbf{y}(t) &= \mathbf{u}\sin 2t + \mathbf{w}\cos 2t = [1,1,0]^T \sin 2t + [0,2,-3]^T\cos 2t \\ &= [\sin 2t, \sin 2t + 2\cos 2t, -3\sin 2t]^T \end{aligned}$$

Ex. 9.5.21: Find general solution of $\begin{cases} x' &= -4x+8y+8z \\ y' &= -4x+4y+2z \\ z' &= 2z \end{cases}$

$$A = \begin{bmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix},\ \det(A-\lambda I) = \begin{vmatrix} -4-\lambda & 8 & 8 \\ -4 & 4-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$\begin{aligned} \Rightarrow p(\lambda) &= (2-\lambda)\begin{vmatrix} -4-\lambda & 8 \\ -4 & 4-\lambda \end{vmatrix} = (2-\lambda)[(-4-\lambda)(4-\lambda)+32] \\ &= (2-\lambda)(\lambda^2+16) \ \Rightarrow\ \lambda_1 = 2,\ \lambda_2 = 4i,\ \lambda_3 = \overline{\lambda_2} \end{aligned}$$

$$A-2I = \begin{bmatrix} -6 & 8 & 8 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & -1/2 \\ -3 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$A-4iI = \begin{bmatrix} -4-4i & 8 & 8 \\ -4 & 4-4i & 2 \\ 0 & 0 & 2-4i \end{bmatrix} \to \begin{bmatrix} 1 & -1+i & 0 \\ -1-i & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1(2,1,1+i),R2(2,3)} \begin{bmatrix} 1 & -1+i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1-i \\ 1 \\ 0 \end{bmatrix}$$

Set $\mathbf{v}_2 = \mathbf{u}_2 + i\mathbf{w}_2,\ \mathbf{u}_2 = [1,1,0]^T,\ \mathbf{w}_2 = [-1,0,0]^T$

General solution: $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t)$, where

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{2t}\mathbf{v}_1 = e^{2t}[0,-1,1]^T \\
\mathbf{x}_2(t) &= \mathbf{u}_2\cos 4t - \mathbf{w}_2 \sin 4t = [1,1,0]^T\cos 4t - [-1,0,0]^T \sin 4t \\
&= [\cos 4t + \sin 4t, \cos 4t, 0]^T \\
\mathbf{x}_3(t) &= \mathbf{u}_2\sin 4t + \mathbf{w}_2 \cos 4t = [1,1,0]^T\sin 4t + [-1,0,0]^T \cos 4t \\
&= [\sin 4t - \cos 4t, \sin 4t, 0]^T
\end{aligned}$$

Ex. 9.5.27: Find solution to IC $\mathbf{x}(0) = [1,0,0]^T$ for system of Ex. 21

F.M.: $$X(t) = \begin{bmatrix} 0 & \cos 4t + \sin 4t & \sin 4t - \cos 4t \\ -e^{2t} & \cos 4t & \sin 4t \\ e^{2t} & 0 & 0 \end{bmatrix}, \; X(0) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Solve $X(0)\mathbf{c} = \mathbf{x}(0)$ for $\mathbf{c}$:

$$[X(0), \mathbf{x}(0)] = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$\Rightarrow$ Solution $\mathbf{x}(t) = -\mathbf{x}_3(t) = [\cos 4t - \sin 4t, -\sin 4t, 0]^T$

Ex. 9.5.37: Find F.S.S. for $\mathbf{y}' = A\mathbf{y}$, $A = \begin{bmatrix} -6 & 2 & -3 \\ -1 & -1 & -1 \\ 4 & -2 & 1 \end{bmatrix}$, using a computer

Use Matlab's *poly*, *roots*, *null* commands with rational format:

```
>> A=[-6 2 -3;-1 -1 -1;4 -2 1];
>> format rat,cpol=poly(A);
>> eigvals=roots(cpol)
eigvals =
     -3
     -2
     -1
```

```
>> v1=null(A+3*eye(3),'r');
>> v2=null(A+2*eye(3),'r');
>> v3=null(A+eye(3),'r');
>> [v1';v2';v3']
ans =
      -1        0          1
     -1/2      1/2         1
      -1       -1          1
```

$\Rightarrow$ eigenvalues and eigenvectors (multiply second eigenvector by 2):

$$\lambda_1 = -3 \quad \leftrightarrow \quad \mathbf{v}_1 = [-1, 0, 1]^T$$
$$\lambda_2 = -2 \quad \leftrightarrow \quad \mathbf{v}_2 = [-1, 1, 2]^T$$
$$\lambda_3 = -1 \quad \leftrightarrow \quad \mathbf{v}_3 = [-1, -1, 1]^T$$

$\Rightarrow$ fundamental set of solutions:

$$\begin{aligned} \mathbf{y}_1(t) &= e^{-3t}[-1, 0, 1]^T \\ \mathbf{y}_2(t) &= e^{-2t}[-1, 1, 2]^T \\ \mathbf{y}_3(t) &= e^{-t}[-1, -1, 1]^T \end{aligned}$$

Ex. 9.5.41: Find F.S.S. for $\mathbf{y}' = A\mathbf{y}$, $A = \begin{bmatrix} -18 & -18 & 10 \\ 18 & 17 & -10 \\ 10 & 10 & -7 \end{bmatrix}$, using a computer

This time use Matlab's symbolic toolbox:

```
>> A=[-18 -18 10;18 17 -10;10 10 -7];
>> poly(sym(A));eigvals=solve(ans)
eigvals =
[     -2]
[ -3+2*i]
[ -3-2*i]
```

```
>> v1=null(sym(A)+2*eye(3));
>> v2=null(sym(A)-eigvals(2)*eye(3));
>> [v1';transpose(v2)]
ans =
[1,          -2,          -2]
[1, -5/4-1/4*i, -3/4-1/4*i]
```

$\Rightarrow$ eigenvalues and eigenvectors (multiply $\mathbf{v}_1$ by -1, $\mathbf{v}_2$ by -4):

$$\lambda_1 = -2 \quad \leftrightarrow \quad \mathbf{v}_1 = [-1, 2, 2]^T$$
$$\lambda_2 = -3 + 2i \quad \leftrightarrow \quad \mathbf{v}_2 = [-4, 5+i, 3+i]^T \;\Rightarrow\; \mathbf{u}_2 = [-4, 5, 3]^T,\ \mathbf{w}_2 = [0, 1, 1]^T$$

$\Rightarrow$ fundamental set of solutions:

$$\begin{aligned}
\mathbf{y}_1(t) &= e^{-2t}[-1, 2, 2]^T \\
\mathbf{y}_2(t) &= e^{-3t}(\mathbf{u}_2 \cos 2t - \mathbf{w}_2 \sin 2t) = e^{-3t}([-4, 5, 3]^T \cos 2t - [0, 1, 1]^T \sin 2t) \\
&= e^{-3t}[-4\cos 2t, 5\cos 2t - \sin 2t, 3\cos 2t - \sin 2t]^T \\
\mathbf{y}_3(t) &= e^{-3t}(\mathbf{u}_2 \sin 2t + \mathbf{w}_2 \cos 2t) = e^{-3t}([-4, 5, 3]^T \sin 2t + [0, 1, 1]^T \cos 2t) \\
&= e^{-3t}[-4\sin 2t, 5\sin 2t + \cos 2t, 3\sin 2t + \cos 2t]^T
\end{aligned}$$

Note: $[5/(-3-i)]\mathbf{v}_2 = [-6+2i, 8-i, 5]^T \rightarrow$ answer given in text.

Ex. 9.5.45: Solve system of Ex. 37 for IC $\mathbf{y}(0) = [-6, 2, 9]^T$ via computer

Let $Y(0) = [\mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_3(0)] = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$. Solve $Y(0)\mathbf{c} = \mathbf{y}(0)$:

```
>> M=[-1 -1 -1 -6;0 1 -1 2;1 2 1 9];
>> rref(M)
ans =
     1     0     0     2
     0     1     0     3
     0     0     1     1
```

$\Rightarrow c_1 = 2,\ c_2 = 3,\ c_3 = 1$

$$\Rightarrow \mathbf{y}(t) = 2e^{-3t}[-1, 0, 1]^T + 3e^{-2t}[-1, 1, 2]^T + e^{-t}[-1, -1, 1]^T$$

or $\mathbf{y}(t) = [-2e^{-3t} - 3e^{-2t} - e^{-t}, 3e^{-2t} - e^{-t}, 2e^{-3t} + 6e^{-2t} + e^{-t}]^T$

Ex. 9.5.49: Solve system of Ex. 41 for IC $\mathbf{y}(0) = [-1, 7, 3]^T$ via computer

$Y(0) = \begin{bmatrix} -1 & -4 & 0 \\ 2 & 5 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $Y(0)\mathbf{c} = \mathbf{y}(0)$

```
>> M=[-1 -4 0 -1;2 5 1 7;2 3 1 3];
>> rref(M)
ans =
     1     0     0    -7
     0     1     0     2
     0     0     1    11
```

$$\mathbf{y}(t) = -7e^{-2t}[-1, 2, 2]^T + 2e^{-3t}[-4\cos 2t, 5\cos 2t - \sin 2t, 3\cos 2t - \sin 2t]^T + 11e^{-3t}[-4\sin 2t, 5\sin 2t + \cos 2t, 3\sin 2t + \cos 2t]^T$$

$$y_1(t) = 7e^{-2t} - 4e^{-3t}(2\cos 2t + 11\sin 2t)$$
$$y_2(t) = -14e^{-2t} + e^{-3t}(21\cos 2t + 53\sin 2t)$$
$$y_3(t) = -14e^{-2t} + e^{-3t}(17\cos 2t + 31\sin 2t)$$