Ch. 9: Constant Coefficients Linear Systems

9.1 Overview of Technique: Eigenvalues/Eigenvectors

Homogeneous system:

$$\mathbf{x}' = A\mathbf{x} \qquad (1)$$

A: constant $n \times n$-matrix

If $n = 1$: $x' = ax \Rightarrow x(t) = Ce^{at}$

Try exponential form for (1):

$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ ($\mathbf{v}$: constant vector)

Sub $\mathbf{x}(t)$ in (1) $\Rightarrow$

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\mathbf{v} = A\mathbf{x}(t) = Ae^{\lambda t}\mathbf{v}$$

$$\Rightarrow \ \lambda\mathbf{v} = A\mathbf{v}$$

Def.: A number λ is an eigenvalue of A if there is a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$A\mathbf{v} = \lambda\mathbf{v} \qquad (2)$$

If λ is an eigenvalue, then any $\mathbf{v} \neq \mathbf{0}$ satisfying (2) is called an eigenvector for λ.

Thm.: If λ is eigenvalue of A and $\mathbf{v}$ is eigenvector for λ, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution of (1).

Characteristic Polynomial

Rewrite (2) (using $\mathbf{v} = I\mathbf{v}$):

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

Since $\mathbf{v} \neq \mathbf{0} \Rightarrow \det(A - \lambda I) = 0$

Def.: $p(\lambda) = \det(A - \lambda I)$ = characteristic polynomial

Note: the degree of $p(\lambda)$ is n.

$\Rightarrow p(\lambda)$ has n roots (if counted with multiplicities)

Thm.: The eigenvalues of A are the roots of

$$p(\lambda) = \det(A - \lambda I) = 0 \qquad (3)$$

If λ is a root of (3), then any $\mathbf{v} \neq \mathbf{0}$ in $\text{null}(A - \lambda I)$ is an eigenvector for λ.

Def.: If λ is an eigenvalue of A, then $\text{null}(A - \lambda I)$ is called the eigenspace of λ.

Thm.: Eigenvectors for distinct eigenvalues are linearly independent.

Consequence:

If $p(\lambda)$ has n distinct real roots

$$\lambda_1, \ldots, \lambda_n$$

then A has n linearly independent eigenvectors

$$\mathbf{v}_1, \ldots, \mathbf{v}_n$$

$$\Rightarrow \; e^{\lambda_1 t}\mathbf{v}_1, \ldots, e^{\lambda_n t}\mathbf{v}_n$$

is fundamental set of solutions.

Complications:

- complex eigenvalues (9.5)
- repeated roots (9.6)

2d Systems: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
(Sec. 9.2-4)

$$\begin{aligned} p(\lambda) &= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) \end{aligned}$$

Set $\quad T = a + d \quad$ (trace of A)
$\qquad D = ad - bc \quad$ ($\det(A)$)

$$\Rightarrow \; p(\lambda) = \lambda^2 - T\lambda + D$$

Roots of $p(\lambda)$:

$$\lambda_{1,2} = \left(T \pm \sqrt{T^2 - 4D}\,\right)/2$$

Roots are real and distinct if

$$T^2 - 4D > 0$$

Eigenvector Solutions of 2d Systems for Real Eigenvalues

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \left\{ \begin{array}{rcl} T & = & a+d \\ D & = & ad-bc \end{array} \right\}$$

$$p(\lambda) = \lambda^2 - T\lambda + D$$

Assume $T^2 - 4D > 0$

$\Rightarrow$ A has two distinct real eigenvalues $\lambda_{1,2}$

Let $\mathbf{v}_1 \neq \mathbf{0}$ be in null$(A - \lambda_1 I)$
$\mathbf{v}_2 \neq \mathbf{0}$ be in null$(A - \lambda_2 I)$

$\mathbf{v}_1, \mathbf{v}_2$ are linearly independent
$\Rightarrow$ Fundamental Solution Set:

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1, \;\; \mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$$

Fundamental Matrix:

$$X(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t)]$$

General Solution:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = X(t)\mathbf{c}$$

Ex.: $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} \Rightarrow T = 1,\ D = -2$

$\Rightarrow\ p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

$\Rightarrow$ Eigenvalues: $\lambda_1 = 2,\ \lambda_2 = -1$

$$A - 2I = \begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix}, (A - 2I)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A + I = \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix},\ (A + I)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ eigenvectors $\left\{ \begin{array}{l} \mathbf{v}_1 = [1, 1]^T \\ \mathbf{v}_2 = [2, 1]^T \end{array} \right\}$

$$\Rightarrow \mathbf{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x}_2(t) = e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are a fundamental set of solutions.
Fundamental matrix:

$$X(t) = \begin{bmatrix} e^{2t} & 2e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}$$

General Solution:

$$\mathbf{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = X(t)\mathbf{c}$$

Worked Out Examples from Exercises

Ex. 9.1.5: Find $p(\lambda)$ and eigenvalues "by hand" for $A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$

This is a 2×2-matrix with $T = 1$, $D = -2 \Rightarrow p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$
$\Rightarrow$ Eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$.

Ex. 9.1.11: Find $p(\lambda)$ and eigenvalues "by hand" for $A = \begin{bmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{bmatrix}$

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -4 & -2 \\ 0 & 1-\lambda & 1 \\ -6 & -12 & 2-\lambda \end{vmatrix} \\
&= (-1)^{2+2}(1-\lambda)\begin{vmatrix} -1-\lambda & -2 \\ -6 & 2-\lambda \end{vmatrix} + (-1)^{2+3}1\begin{vmatrix} -1-\lambda & -4 \\ -6 & -12 \end{vmatrix} \\
&= (1-\lambda)[(-1-\lambda)(2-\lambda) - 12] - [(-1-\lambda)(-12) - 24] \\
&= -(1-\lambda)(-\lambda^2 + \lambda + 14) + 12(1-\lambda) \\
&= (1-\lambda)(\lambda^2 - \lambda - 2) \\
&= (1-\lambda)(\lambda + 1)(\lambda - 2)
\end{aligned}$$

$\Rightarrow$ Eigenvalues $1, -1, 2$

Ex. 9.1.27: Find fundamental solution set "by hand" for $\mathbf{y}' = A\mathbf{y}$ if

$$A = \begin{bmatrix} -3 & 0 & 2 \\ 6 & 3 & -12 \\ 2 & 2 & -6 \end{bmatrix}$$

$$\begin{aligned}
p(\lambda) &= \det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 0 & 2 \\ 6 & 3-\lambda & -12 \\ 2 & 2 & -6-\lambda \end{vmatrix} \\
&= (-1)^{1+1}(-3-\lambda)\begin{vmatrix} 3-\lambda & -12 \\ 2 & -6-\lambda \end{vmatrix} + (-1)^{1+3}2\begin{vmatrix} 6 & 3-\lambda \\ 2 & 2 \end{vmatrix} \\
&= -(3+\lambda)[(3-\lambda)(-6-\lambda)+24] + 2[12-2(3-\lambda)] \\
&= -(3+\lambda)(\lambda^2+3\lambda+6) + 4(\lambda+3) = -(\lambda+3)(\lambda^2+3\lambda+2) \\
&= -(\lambda+3)(\lambda+1)(\lambda+2)
\end{aligned}$$

$\Rightarrow$ eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$. Find eigenvectors:

1. $\lambda_1 = -1$:

$$A + I = \begin{bmatrix} -2 & 0 & 2 \\ 6 & 4 & -12 \\ 2 & 2 & -5 \end{bmatrix} \xrightarrow{R3(1,-1/2)} \begin{bmatrix} 1 & 0 & -1 \\ 6 & 4 & -12 \\ 2 & 2 & -5 \end{bmatrix}$$

$$\xrightarrow{R1(2,1,-6),R1(3,1,-2)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & -6 \\ 0 & 2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Set free variable $y_3 = 2 \Rightarrow y_2 = 3$, $y_1 = 2 \Rightarrow$ eigenvector $\mathbf{v}_1 = [2,3,2]^T$.

2. $\lambda_2 = -2$:

$$A + 2I = \begin{bmatrix} -1 & 0 & 2 \\ 6 & 5 & -12 \\ 2 & 2 & -4 \end{bmatrix} \xrightarrow{R1(2,1,6),R1(3,1,2)} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Set $y_3 = 1 \Rightarrow y_2 = 0$, $y_1 = 2 \Rightarrow$ eigenvector $\mathrm{v}_2 = [2, 0, 1]^T$

3. $\lambda_3 = -3$:

$$A + 3I = \begin{bmatrix} 0 & 0 & 2 \\ 6 & 6 & -12 \\ 2 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Set free variable $y_2 = 1 \Rightarrow y_3 = 0$, $y_1 = -1 \Rightarrow$ eigenvector $\mathbf{v}_3 = [-1, 1, 0]^T$.

$\Rightarrow$ **fundamental solution set:**

$$\mathbf{y}_1(t) = e^{-t} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \mathbf{y}_2(t) = e^{-2t} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{y}_3(t) = e^{-3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Note: Associated fundamental matrix is $Y(t) = \begin{bmatrix} 2e^{-t} & 2e^{-2t} & -e^{-3t} \\ 3e^{-t} & 0 & e^{-3t} \\ 2e^{-t} & e^{-2t} & 0 \end{bmatrix}$

General solution: $\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + c_3\mathbf{y}_3(t) = Y(t)\mathbf{c}$; $\mathbf{c} = [c_1, c_2, c_3]^T$

Ex. 9.1.36: Find eigenvalues and eigenvectors using a computer for

$$A = \begin{bmatrix} -6 & 5 & -9 & 10 \\ 10 & -7 & 13 & -16 \\ 4 & -4 & 8 & -8 \\ -5 & 3 & -5 & 7 \end{bmatrix}$$

1. Numerical computation via Matlab's *poly*, *roots*, and *null* commands:

```
>> A=[-6 5 -9 10;10 -7 13 -16;4 -4 8 -8;-5 3 -5 7];cpol=poly(A)
cpol =
   1.0000   -2.0000   -1.0000    2.0000   -0.0000
```

The output of *poly* is a row vector whose entries are approximated values for the coefficients of the characteristic polynomial:

$$p(\lambda) \approx 1.0000 \times \lambda^4 - 2.0000 \times \lambda^3 - 1.0000 \times \lambda^2 + 2.0000 \times \lambda - 0.0000$$

Find the roots of the characteristic polynomial:

```
>> evals=roots(cpol)
evals =
   -1.0000
    2.0000
    1.0000
    0.0000
```

So the eigenvalues (roots of $p(\lambda)$) are approximately -1.0000, 2.0000, 1.0000, 0.0000.

They can be accessed via *evals(1)*, *evals(2)* etc.

Now compute bases for the nullspaces of the eigenvalues using the *null*–command:

```
>> v1=null(A-evals(1)*eye(4))
 v1 =
   -0.5774
    0.5774
    0.0000
   -0.5774
```

(The $n \times n$ identity matrix is denoted in Matlab by *eye(n)* – here $n = 4$.) Analogously one can compute the other three eigenvectors.

2. Symbolic computation using Matlab's *poly*, *factor* or *solve*, and *null* commands:

poly and *null* work also for symbolically defined matrices. The *roots* command works only for numerically defined vectors. To find roots of a symbolically defined polynomial, use the commands *factor* or *solve*.

```
>> sym_A=sym(A);sym_cpol=poly(sym_A)
sym_cpol =
x^4-2*x^3-x^2+2*x
```

Note that here the output is a symbolic polynomial expression with (default) variable x.

You can find the eigenvalues with the *factor* command:

```
>> factor(sym_cpol)
ans =
x*(x-1)*(x-2)*(x+1)
```

So the exact eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = -1$.
Alternatively you can find them using *solve*:

```
>> sym_evals=solve(sym_cpol)
sym_evals =
[  0]
[  1]
[  2]
[ -1]
```

Now find eigenvectors:

```
>> sym_v1=null(sym_A-sym_evals(1)*eye(4))
sym_v1 =
[ 1]
[ 1]
[ 1]
[ 1]
```

hence $\mathbf{v}_1 = [1, 1, 1, 1]^T$. Analogously one finds the eigenvectors for λ_2, $\lambda_3, , \lambda_4$:

$$\mathbf{v}_2 = [0, -2, 0, 1]^T,\ \mathbf{v}_3 = [-1, 0, 2, 1]^T,\ \mathbf{v}_4 = [1, -1, 0, 1]^T.$$

Ex. 9.1.29: Find eigenvalues and eigenvectors using a computer for

$$A = \begin{bmatrix} -7 & 2 & 10 \\ 0 & 1 & 0 \\ -5 & 2 & 8 \end{bmatrix}.$$

Eigenvalues and eigenvectors can be computed directly in Matlab with the *eig* command. Outputs:

V: matrix whose columns are eigenvectors

D: diagonal matrix whose diagonal entries are eigenvalues

Without specification, outputs are floating point numbers:

```
A=[-7 2 10;0 1 0;-5 2 8];
[V,D]=eig(A)
V =
   -0.8944   -0.7071   -0.5774
         0         0    0.5774
   -0.4472   -0.7071   -0.5774
D =
    -2     0     0
     0     3     0
     0     0     1
```

Symbolic computation yields exact values if available:

```
A=[-7 2 10;0 1 0;-5 2 8];
[V,D]=eig(sym(A))
V =
[  1,  2,  1]
[ -1,  0,  0]
[  1,  1,  1]
D =
[  1,  0,  0]
[  0, -2,  0]
[  0,  0,  3]
```

Hence $\lambda_1 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\lambda_2 = -2$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\lambda_3 = 3$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Ex. 9.1.39: Find fundamental solution set via computer for $\mathbf{y}' = A\mathbf{y}$ if

$$A = \begin{bmatrix} 20 & -34 & -10 \\ 12 & -21 & -5 \\ -2 & 4 & -2 \end{bmatrix}$$

Editing A in Matlab and applying Matlab's *eig* command to *sym(A)* yields the following eigenvalues and eigenvectors:

$$\lambda_1 = -4,\, \mathbf{v}_1 = [-1,-1,1]^T, \;\; \lambda_2 = -2,\, \mathbf{v}_2 = [2,1,1]^T, \;\; \lambda_3 = 3,\, \mathbf{v}_3 = [2,1,0]^T$$

$\Rightarrow$ fundamental solution set:

$$\mathbf{y}_1(t) = e^{-4t}[-1,-1,1]^T, \;\; \mathbf{y}_2(t) = e^{-2t}[2,1,1]^T, \;\; \mathbf{y}_3(t) = e^{3t}[2,1,0]^T$$

Ex. 9.1.49(i): Find determinant and eigenvalues of $A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$ via computer.

Describe any relationship between eigenvalues and determinant.

No computer necessary to find $\det(A) = -4$.
Eigenvalues (using Matlab): $\lambda_1 = 2$, $\lambda_2 = -2$, hence $\lambda_1\lambda_2 = -4 = \det(A)$.

Ex. 9.1.51(i): Find eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}$ via computer.

Describe any relationship between eigenvalues and triangular structure of A.

Matlab $\rightarrow$ eigenvalues $\lambda_1 = 2$, $\lambda_2 = -4$. These are the diagonal entries of A.

Thm.: The eigenvalues of a lower or upper triangular matrix are the diagonal entries.

Ex. 9.2.3: Find general solution of $\mathbf{y}' = A\mathbf{y}$ for $A = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix}$

$T = -7$, $D = 12 \Rightarrow T^2 - 4D = 1 \Rightarrow$ eigenvalues $\lambda_{1,2} = -7/2 \pm 1/2$
$\Rightarrow \lambda_1 = -3$, $\lambda_2 = -4$. Find eigenvectors:

$$A + 3I = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad A + 4I = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Fundamental set of solutions:

$$\mathbf{y}_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{y}_2(t) = e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution:

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) = \begin{bmatrix} e^{-3t} & e^{-4t} \\ 2e^{-3t} & e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Ex. 9.2.9: Find solution of system of Ex. 3 for IC $\mathbf{y}(0) = [0, -1]^T$

Match c_1, c_2 to IC:

$$\mathbf{y}(0) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{y}(t) = -e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-4t} - e^{-3t} \\ e^{-4t} - 2e^{-3t} \end{bmatrix}$$