8.4-5: Linear Systems

General Form:

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + f_2(t) \\ \vdots \quad & \vdots \quad \vdots \\ x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}$$

- $a_{ij}(t), f_i(t)$: known functions on interval I: $\alpha < t < \beta$

Matrix-vector notation:

$$\begin{aligned} \mathbf{x} &= [x_1, \ldots, x_n]^T \\ \mathbf{f}(t) &= [f_1(t), \ldots, f_n(t)]^T \\ A(t) &= [a_{ij}(t)]_{nn} \end{aligned}$$

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t) \qquad (1)$$

- (1) is **homogeneous** if $\mathbf{f}(t) = \mathbf{0}$
- (1) is **nonhomogeneous** if $\mathbf{f}(t) \neq \mathbf{0}$
- (1) has constant coefficients if $a_{ij}(t) = a_{ij}$ are constants

Initial Value Problem:

$$\left.\begin{aligned} \mathbf{x}' &= A(t)\mathbf{x} + \mathbf{f}(t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{aligned}\right\} \quad (2)$$

Thm.: If $a_{ij}(t)$, $f_i(t)$ are continuous on I and $t_0 \in I$, then (2) has a unique solution on I.

Ex.: $\left\{\begin{aligned} x' &= 3x - 5y \\ y' &= -2x \end{aligned}\right\} \Rightarrow$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is hom., constant coefficients.

Ex.: $\left\{\begin{aligned} u' &= \cos(t)v \\ v' &= u + \sin t \end{aligned}\right\}$

is nonhom., non-constant coefs.:

$$A(t) = \begin{bmatrix} 0 & \cos t \\ 1 & 0 \end{bmatrix}, \mathbf{f}(t) = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}$$

Ex.: $x' = xy$, $y' = x$ is nonlinear

Superposition Principle for Homogeneous Systems (8.5)

$$\mathbf{x}' = A(t)\mathbf{x} \qquad (3)$$

$$A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y} \quad \Rightarrow$$

Thm.: (Superposition Principle)

If $\mathbf{x}_1(t), \mathbf{x}_2(t)$ are solutions of (3) and c_1, c_2 are arbitrary constants, then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

is also a solution.

Superposition principle does in general **not** hold for

- nonlinear systems
- nonhomogeneous linear systems

Ex.: $x' = x^2 \rightarrow$ solution $x(t) = -1/t$. $y(t) = -x(t) = 1/t$ is *not* solution, because $y' = -1/t^2$, $y^2 = 1/t^2$.

Ex.: $x' = x - 1 \rightarrow$ solution $x(t) = 1$. $y(t) = 0 \cdot x(t) = 0$ is *not* solution.

Ex.: $\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \ \mathbf{x}_2(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

are solutions (verify by substitution)

$$\Rightarrow \mathbf{x}(t) = c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

is solution for any c_1, c_2. Rewrite:

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix} \mathbf{c}, \ \ \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Consider IC: $\mathbf{x}(0) = \mathbf{x}_0 \Rightarrow$

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{c} = \mathbf{x}_0$$

Invert matrix: $\mathbf{c} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{x}_0$

$\Rightarrow$ unique solution for *any* $\mathbf{x}_0$

e.g.: $\mathbf{x}_0 = [3, -1]^T \Rightarrow \mathbf{c} = [2, 1]^T \Rightarrow$

$$\mathbf{x}(t) = 2\mathbf{x}_1(t) + \mathbf{x}_2(t) = \begin{bmatrix} 2e^{-t} + e^{3t} \\ -2e^{-t} + e^{3t} \end{bmatrix}$$

Linear Independence, Fundamental Set of Solutions (8.5)

Basic Existence and Uniqueness Theorem $\Rightarrow$

Thm.: Assume $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are k solutions of

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^n \tag{3}$$

for t on I and that $a_{ij}(t)$ are continuous on I. Let $t_0 \in I$.

1. If there are constants $c_1, \ldots, c_k$, not all 0, such that $c_1\mathbf{x}_1(t_0) + \cdots + c_k\mathbf{x}_k(t_0) = \mathbf{0}$, then $c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t) \equiv \mathbf{0}$.

2. If the vectors $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_k(t_0)$ are linearly independent, then $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are linearly independent for any t on I.

Def.: (Fundamental Set) Assume $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are n solutions of (3) on an interval I on which $A(t)$ is continuous. $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are called a fundamental set of solutions if the vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly independent for all t on I.

Note: Thm. $\Rightarrow$ it is sufficient that $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)$ are linearly independent for some t_0.

Solution Strategy (8.5)

- Find fundamental set $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ (Ch. 9)
- General solution: $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$
- Rewrite this as

$$\mathbf{x}(t) = X(t)\mathbf{c}$$

$$\begin{aligned} \mathbf{c} &= [c_1, \ldots, c_n]^T \\ X(t) &= [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)] \end{aligned}$$

- $X(t)$ $(n \times n)$ is called **fundamental matrix**
- Match $\mathbf{c}$ to IC:

$$\mathbf{x}(t_0) = X(t_0)\mathbf{c} = \mathbf{x}_0$$

$$\Rightarrow \mathbf{c} = \Big(X(t_0)\Big)^{-1}\mathbf{x}_0$$

Wronskian: $W(t) = \det(X(t))$

Condition for linear independence: $W(t_0) \neq 0$

Nonhomogeneous System:

Given a particular solution $\mathbf{x}_p(t)$, any solution $\mathbf{x}(t)$ of

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$$

can be written in the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + X(t)\mathbf{c}$$

Ex.: $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{x}$. Solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \ \mathbf{x}_2(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{bmatrix}$$

Wronskian of $\mathbf{x}_1(t), \mathbf{x}_2(t)$:

$$\begin{aligned} W(t) &= \det(X(t)) \\ &= e^{-t}e^{3t} + e^{-t}e^{3t} = 2e^{2t} \neq 0 \end{aligned}$$

$\Rightarrow X(t)$ is fundamental matrix.

Worked Out Examples from Exercises

Ex. 8.4.13: If possible, place system in form (1), if not possible explain why.

$$\left\{\begin{array}{rcl} x_1' & = & -2x_1 + x_2^2 \\ x_2' & = & 3x_1 - x_2 \end{array}\right\}$$ cannot be placed because it is nonlinear.

Ex. 8.4.14: Same as Ex. 8.4.13

$$\left\{\begin{array}{rcl} x_1' & = & -2x_1 + 3tx_2 + \cos t \\ tx_2' & = & x_1 - 4tx_2 + \sin t \end{array}\right\} \to \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 3t \\ 1/t & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos t \\ (\sin t)/t \end{bmatrix}$$

Ex. 8.5.4: Rewrite system using matrix notation

$$\left\{\begin{array}{rcl} x_1' & = & -x_2 \\ x_2' & = & x_1 \end{array}\right\} \to \mathbf{x}' = A\mathbf{x} \text{ with } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Ex. 8.5.6: Rewrite system using matrix notation

$$\left\{\begin{array}{rcl} x_1' & = & -x_2 + \sin t \\ x_2' & = & x_1 \end{array}\right\} \to \mathbf{x}' = A\mathbf{x} + \mathbf{f}(t) \text{ with } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} \sin t \\ 0 \end{bmatrix}$$

Ex. 8.5.10: Let $\mathbf{x}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, $\mathbf{y}(t) = \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$

Show that $\mathbf{x}(t), \mathbf{y}(t)$ are solutions of the system of Ex. 8.5.4.
Verify that any linear combination is a solution.

1. $\mathbf{x}(t) \to x_1(t) = \cos t$, $x_2(t) = \sin t$. $x_1' = -\sin t = -x_2$, $x_2' = \cos t = x_1$: OK.
2. $\mathbf{y}(t) \to y_1(t) = \sin t$, $y_2(t) = -\cos t$. $y_1' = \cos t = -y_2$, $y_2' = \sin t = y_1$: OK.
3. $(c_1\mathbf{x}(t) + c_2\mathbf{y}(t))' = c_1\mathbf{x}'(t) + c_2\mathbf{y}'(t) = c_1 A\mathbf{x}(t) + c_2 A\mathbf{y}(t) = A(c_1\mathbf{x}(t) + c_2\mathbf{y}(t))$

Ex. 8.5.12: Let $\mathbf{x}_p(t) = \frac{1}{2}\begin{bmatrix} t\sin t - \cos t \\ -t\cos t \end{bmatrix}$

Show that $\mathbf{x}_p(t)$ is a solution of the system of Ex. 8.5.6. Further show that $\mathbf{z}(t) = \mathbf{x}_p(t) + c_1\mathbf{x}(t) + c_2\mathbf{y}(t)$ is also solution, where $\mathbf{x}(t), \mathbf{y}(t)$ are from Ex. 8.5.10.

1. $\mathbf{x}_p(t) \rightarrow x_1(t) = (t\sin t - \cos t)/2$, $x_2(t) = -(t\cos t)/2$.

   a: $x_1'(t) = (t\cos t + \sin t)/2 + (\sin t)/2 = (t\cos t)/2 + \sin t$
   $-x_2(t) + \sin t = (t\cos t)/2 + \sin t$: OK

   b: $x_2'(t) = -(\cos t)/2 + (t\sin t)/2 = x_1(t)$: OK

2. $\mathbf{z}'(t) = \mathbf{x}_p'(t) + c_1\mathbf{x}'(t) + c_2\mathbf{y}'(t) = (A\mathbf{x}_p(t) + \mathbf{f}(t)) + c_1A\mathbf{x}(t) + c_2A\mathbf{y}(t)$
   $= A(\mathbf{x}_p(t) + c_1\mathbf{x}(t) + c_2\mathbf{y}(t)) + \mathbf{f}(t)) = A\mathbf{z}(t) + \mathbf{f}(t)$

Ex. 8.5.18: Let $\mathbf{y}_1(t) = \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix}$, $\mathbf{y}_2(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$

Suppose that $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$ are solutions of a homogeneous linear system. Further suppose that $\mathbf{x}(t)$ is a solution of the same system with IC $\mathbf{x}(0) = [1, -1]^T$. Find c_1, c_2 such that $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

$$\text{Let } Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] \Rightarrow Y(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{c} = (Y(0))^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \Rightarrow c_1 = 2,\ c_2 = -3$$

Ex. 8.5.19: Let $\mathbf{y}_1(t) = \begin{bmatrix} -e^{-t} \\ -e^{-t} \\ e^{-t} \end{bmatrix}$, $\mathbf{y}_2(t) = \begin{bmatrix} 0 \\ -e^{t} \\ 2e^{t} \end{bmatrix}$, $\mathbf{y}_3(t) = \begin{bmatrix} e^{2t} \\ 0 \\ 2e^{2t} \end{bmatrix}$

$\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)$ are solutions of a homogeneous linear system. Check linear dependence or independence of these solutions.

Let $Y(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)]$. It is sufficient to check for $t = 0$. Wronskian:

$$\begin{aligned} W(0) &= \det(Y(0)) = \begin{vmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} \\ &= (-1)^{2+1}(-1)\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} + (-1)^{2+2}(-1)\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -2 + 3 = 1 \end{aligned}$$

$\Rightarrow \mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t)$ are linearly independent for all t.

Confirm this using Matlab's symbolic toolbox:

```
syms t;y1=[-exp(-t);-exp(-t);exp(-t)];y2=[0;-exp(t);2*exp(t)];
y3=[exp(2*t);0;2*exp(2*t)];Y=[y1 y2 y3];simplify(det(Y))
```

Answer in Command Window:

```
>> exp(2*t)
```

Hence $W(t) = e^{2t}$ which is indeed nonzero for all t.