8.2-3: Geometric Interpretation and Qualitative Analysis

Autonomous system:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} = [x_1, \ldots, x_n]^T$$

For any t: $\mathbf{x}(t) \in \mathbf{R}^n$

- $\mathbf{R}^n$: **phase space** ($n = 2$: phase plane)
- **Trajectory:** Curve

  $$\{\mathbf{x}(t) \,|\, t \in I\} \text{ in } \mathbf{R}^n$$

  I: interval on which $\mathbf{x}(t)$ is defined
- **Tangent vectors:**

  $$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$$
- **Vector field:** $\mathbf{x} \to \mathbf{f}(\mathbf{x})$
- $\mathbf{x}(t)$ solution $\Rightarrow$ $\mathbf{x}(t - t_0)$ solution: *same trajectory!*
- If existence and uniqueness, trajectories don't intersect

Example: Lotka-Volterra's predator-prey equations

$$\begin{aligned} R' &= (a - bF)R \\ F' &= (-c + dR)F \end{aligned} \qquad (1)$$

$$a, b, c, d > 0$$

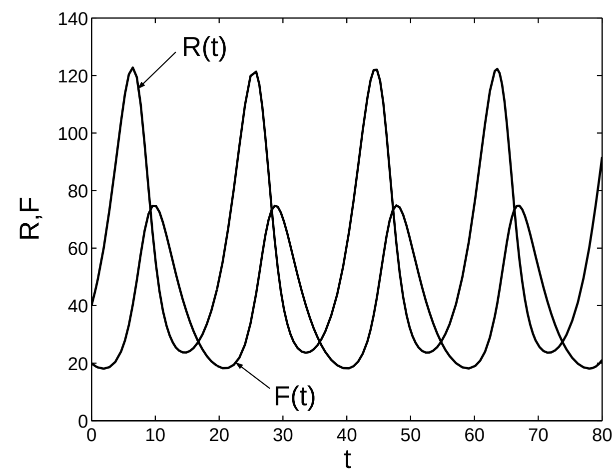
R: number of rabbits
F: number of foxes

Parameters:

$$a = 0.4,\ b = 0.01$$
$$c = 0.3,\ d = 0.005 \qquad (2)$$

IC: $R(0) = 40,\ F(0) = 20$

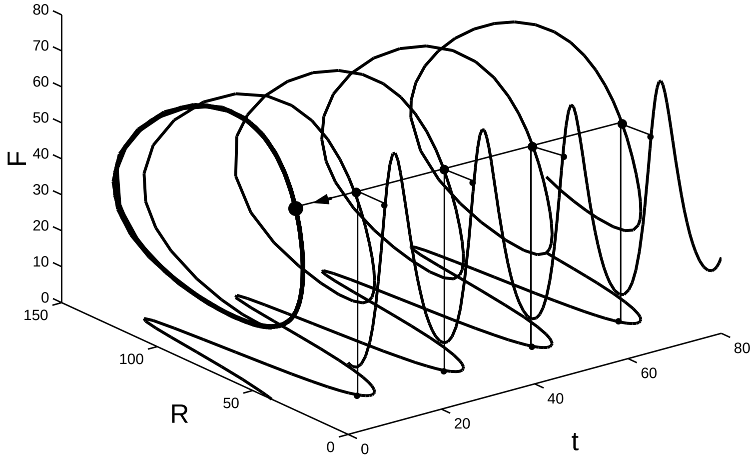
Numerical Solution:



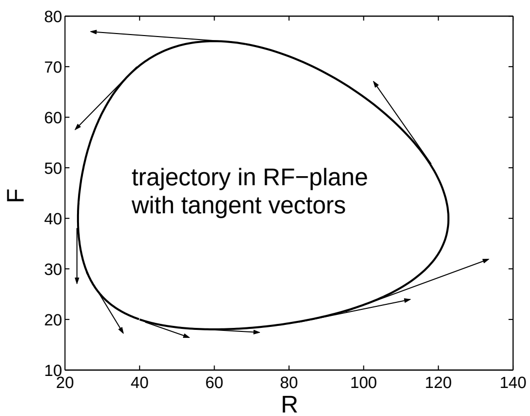
3d and 2d Plots for Eqs. (1,2)

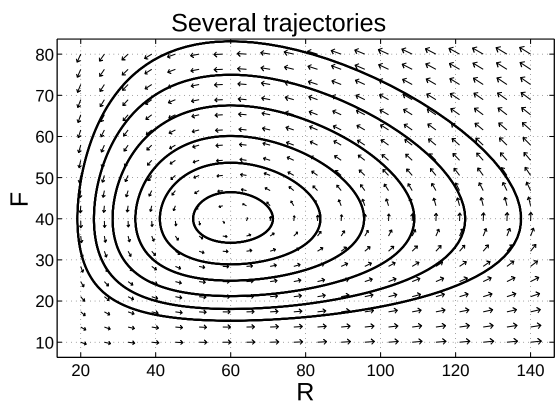
Composite graph

$R(t)$,$F(t)$, 3d curve $(t, R(t), F(t))$, and trajectory



2d trajectories and vector field

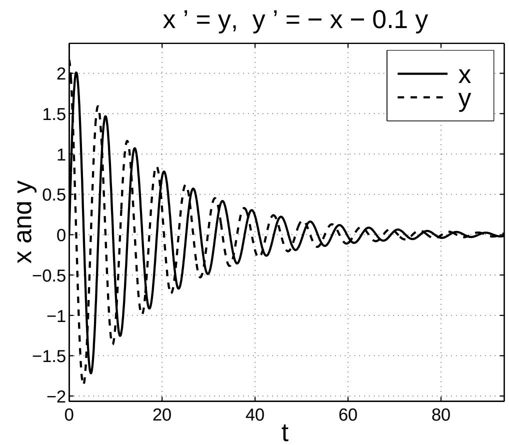




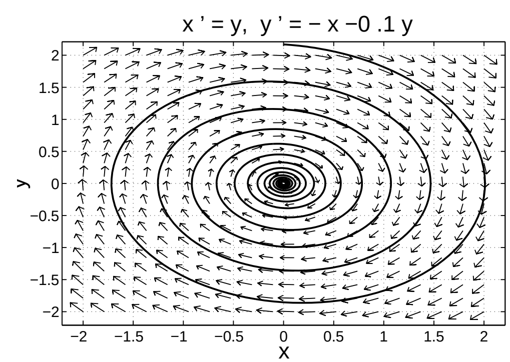
Ex.:
$$\begin{aligned} x' &= y \\ y' &= -x - 0.1y \end{aligned}$$

$x(0) = 0$, $y(0) = 2$

time plots



trajectory and vector field



Equilibrium Points and Nullclines (8.3)

Ex. : $$\begin{aligned} R' &= (a - bF)R \\ F' &= (-c + dR)F \end{aligned}$$

Equilibrium points: $R' = F' = 0$

$$\Rightarrow \begin{cases} (a - bF)R = 0 \\ (-c + dR)F = 0 \end{cases}$$

Solutions:

$[R, F]^T = [0, 0]^T$, $[R, F]^T = [c/d, a/b]^T$

Equilibrium points $\rightarrow$
constant solutions of ODE-system:

$$[R(t), F(t)]^T = [c/d, a/b]^T$$

R-nullcline: $R' = 0$

$$\Rightarrow \quad R = 0 \text{ and } F = a/b$$

F-nullcline: $F' = 0$

$$\Rightarrow \quad F = 0 \text{ and } R = c/d$$

Equilibrium points are intersections of nullclines

Ex.: $$\begin{aligned} x' &= (1 - x - y)x \\ y' &= (4 - 2x - 7y)y \end{aligned}$$

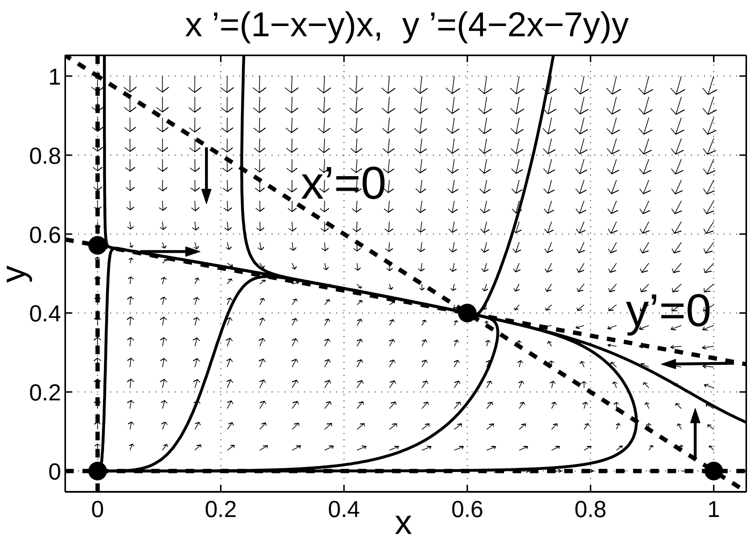
x-nullclines: $x = 0$, $x + y = 1$

y-nullclines: $y = 0$, $2x + 7y = 4$

Equlibrium points:

$(0, 0)$, $(0, 4/7)$, $(1, 0)$, $(3/5, 2/5)$

several solutions, nullclines, and equilibrium points (using *pplane6*)



Worked Out Examples from Exercises

Ex. 8.2.1: Plot (i) $x_1(t)$,$x_2(t)$ and (ii) the curve $t \to (x_1(t), x_2(t))$ for

$$\mathbf{x}(t) = [2e^t - e^{-t}, e^{-t}]^T, \text{ i.e. } x_1(t) = 2e^t - e^{-t},\ x_2(t) = e^{-t}$$

Matlab commands:

```
t=linspace(0,2,100);x1=2*exp(t)-exp(-t);x2=exp(-t);
figure(1),plot(t,x1,'k',t,x2,'k--'),xlabel('t'),ylabel('x_1 and x_2')
figure(2),plot(x1,x2,'k'),xlabel('x_1'),ylabel('x_2'),axis([0 15 0 1])
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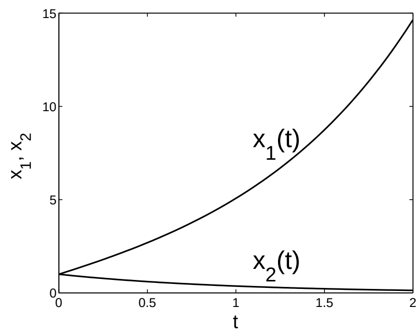
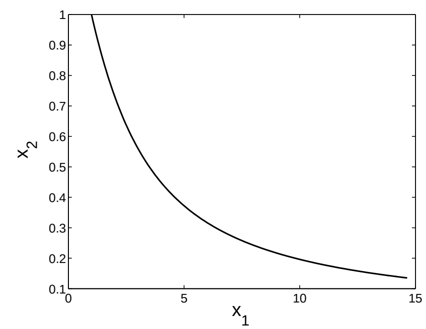
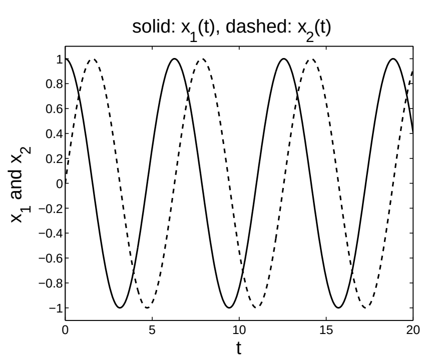
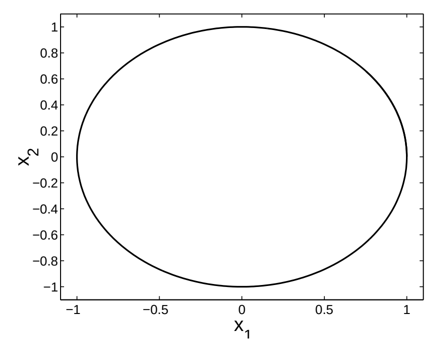




Ex. 8.2.3: Same as Ex. 8.2.1 for

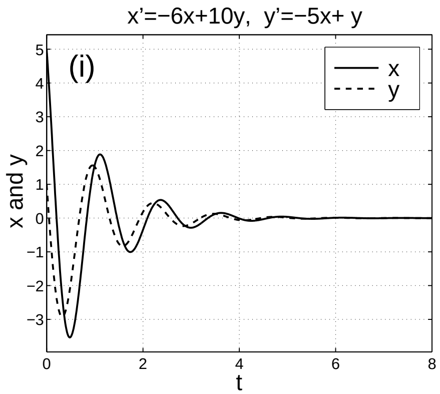
$$\mathbf{x}(t) = [\cos t, \sin t]^T, \text{ i.e. } x_1(t) = \cos t,\ x_2(t) = \sin t$$

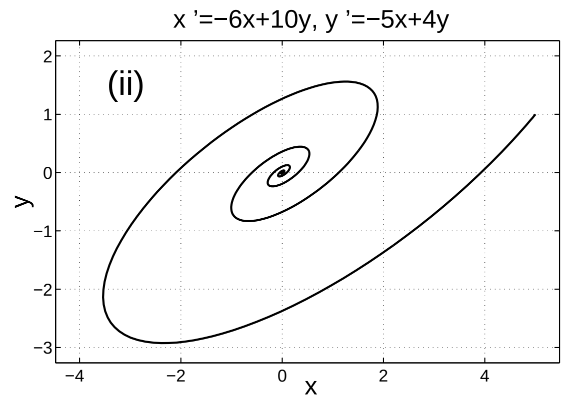




Ex. 8.2.17: Plot (i) solutions $x(t), y(t)$ of IVP as functions of t, (ii) trajectory

IVP: $x' = -6x + 10y$, $y' = -5x + 4y$, $x(0) = 5$, $y(0) = 1$. Use *pplane6*:

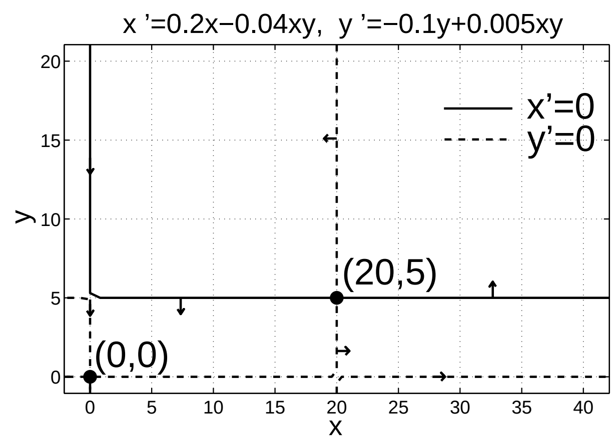




Ex. 8.3.1: Plot (i) nullclines and (ii) equilibrium points for

$$\left\{ \begin{array}{lcl} x' & = & 0.2x - 0.04xy \\ y' & = & -0.1y + 0.005xy \end{array} \right\}. \text{ Nullclines: } \left\{ \begin{array}{lcl} x' = 0 & \Rightarrow & x = 0 \text{ and } y = 5 \\ y' = 0 & \Rightarrow & y = 0 \text{ and } x = 20 \end{array} \right\}$$

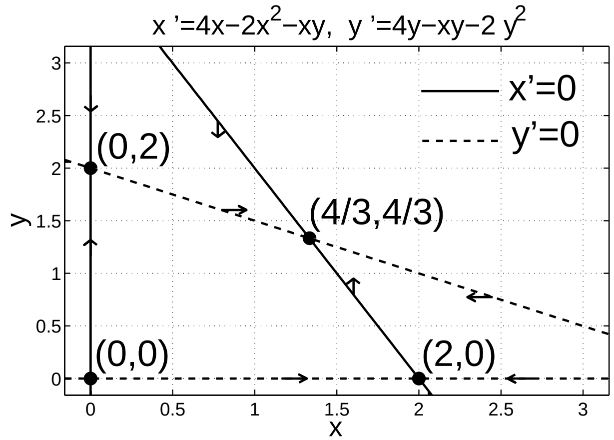
Equilibria: $\left\{ \begin{array}{c} [0,0]^T \\ [20,5]^T \end{array} \right\}$ Use *pplane6*:



Ex. 8.3.2: Plot (i) nullclines and (ii) equilibrium points for

$$\left\{\begin{array}{rcl} x' &=& 4x-2x^2-xy \\ y' &=& 4y-xy-2y^2 \end{array}\right\}. \text{ Nullclines: } \left\{\begin{array}{lcl} x'=0 &\Rightarrow& x=0 \text{ and } 2x+y=4 \\ y'=0 &\Rightarrow& y=0 \text{ and } x+2y=4 \end{array}\right\}$$

Equilibria: $\left\{\begin{array}{cc} [0,0]^T & [2,0]^T \\ [4/3,4/3]^T & [0,2]^T \end{array}\right\}$. *pplane6*:



Ex. 8.3.7: Consider $\left\{\begin{array}{rcl} x' &=& 1-(y-\sin x)\cos x \\ y' &=& \cos x - y + \sin x \end{array}\right\}$

(a) Show that $x(t)=t$, $y(t)=\sin t$ is solution:

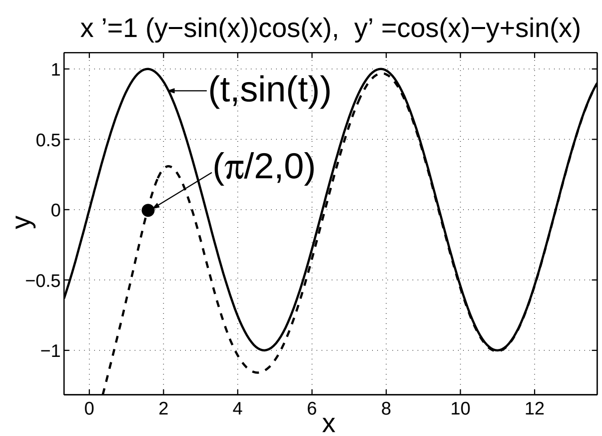
$x'=1$, $\left\{\begin{array}{c} 1-(y-\sin x)\cos x = \\ 1-(\sin t - \sin t)\cos t = 1 \end{array}\right\}$ OK

$y'=\cos t$, $\left\{\begin{array}{c} \cos x - y + \sin x = \\ \cos t - \sin t + \sin t = \cos t \end{array}\right\}$ OK

(c) Show that $y(t)<\sin x(t)$ for all t if $x(0)=\pi/2$, $y(0)=0$:

Solution of (a) satisfies $y=\sin x$. Trajectories don't cross $\Rightarrow y(t)<\sin x(t)$ if $y(0)<\sin x(0)$.

(b) Plot solutions:



Ex. 8.3.12b: Plot the solution of the IVP

$$\left\{ \begin{array}{rcl} R' & = & 0.4R(1-R/100)-0.01RF \\ F' & = & -0.3F+0.005RF \end{array} \right\}, \left\{ \begin{array}{rcl} R(0) & = & 40 \\ F(0) & = & 20 \end{array} \right\}$$

What appears to be the eventual fate of both the predator and prey populations?

Both the "rabbit" and "fox" populations appear to approach equilibrium values $R^* = 60$ and $F^* = 16$, respectively.

Interpretation: In contrast to the original Lotka-Volterra model, equation (1), the rabbits don't grow exponentially if $F = 0$, but approach the carrier capacity $R_c = 100$. This limited growth is not present in the original Lotka-Volterra model. It appears that the introduction of a carrier capacity for the prey can prevent oscillations of the populations.