Chapter 8: Introduction to Systems

8.1 Definitions and Examples

System of 1st order ODEs:

$$\begin{aligned} x_1' &= f_1(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, \ldots, x_n) \end{aligned}$$

Vector notation:

$$\begin{aligned} \mathbf{x} &= [x_1, \ldots, x_n]^T \\ \mathbf{f} &= [f_1, \ldots, f_n]^T \\ \mathbf{x}' &= [x_1', \ldots, x_n']^T \end{aligned}$$

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \qquad (1)$$

n: dimension of system
$n = 2$: planar system

- (1) is autonomous if $\mathbf{f}$ does not depend on t
- (1) is non-autonomous if $\mathbf{f}$ depends on t

Initial Value Problem:

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \qquad (2)$$

Thm.: If $\mathbf{f}$ is continuous in a region R and has continuous partial derivatives $\partial f_i/\partial x_j$ in R, (2) has a unique solution in R.

Ex.:

$$\begin{aligned} x_1' &= -a x_1 x_2 \\ x_2' &= a x_1 x_2 - b x_2 \\ x_3' &= b x_2 \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= [x_1, x_2, x_3]^T \\ \mathbf{f}(\mathbf{x}) &= [-a x_1 x_2, a x_1 x_2 - b x_2, b x_2]^T \end{aligned}$$

$\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is $3d$ autonomous system

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Ex.:

$$\begin{aligned} x' &= v \\ v' &= -x - 0.2v + 2\cos t \end{aligned}$$

is $2d$ non-autonomous system

Reduction of Higher Order Equations

Thm. Any system of higher order ODEs in explicit form can be transformed to a (larger) system of first order ODEs.

Ex.: $x''' + xx'' = \cos t$ (3)

Set $x_1 = x$, $x_2 = x'$, $x_3 = x''$

$$\Rightarrow \begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = x_3 \\ x_3' &= x''' = -xx'' + \cos t \\ &= -x_1x_3 + \cos t \end{aligned}$$

Hence equivalent system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -x_1x_3 + \cos t \end{aligned} \quad (4)$$

Given a solution $x(t)$ of (3) $\Rightarrow$ $[x(t), x'(t), x''(t)]^T$ is solution of (4)

Conversely: Given a solution $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ of (4) $\Rightarrow$ $x(t) = x_1(t)$ is a solution of (3)

General Higher Order ODEs:

nth order ODE in explicit form:

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

Set $x_1 = x$, $x_2 = x'$, ... , $x_n = x^{(n-1)}$

$$\Rightarrow \begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = x_3 \\ &\vdots \\ x_{n-1}' &= x^{(n-1)} = x_n \\ x_n' &= x^{(n)} = f(t, x, x', \dots, x^{(n-1)}) \\ &= f(t, x_1, x_2, \dots, x_n) \end{aligned}$$

$\Rightarrow$ equivalent system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, x_2, \dots, x_n) \end{aligned}$$

Worked Out Examples from Exercises

Ex. 1: Is the system autonomous? What is the dimension?

$$\left.\begin{array}{rcl} x' &=& v \\ v' &=& -x - 0.02v + 2\cos t \end{array}\right\} \text{ is non-autonomous } (\cos t)\text{. Dimension: 2}$$

Ex. 2: Same questions as in Ex. 1

$$\left.\begin{array}{rcl} \theta' &=& \omega \\ \omega' &=& -(g/L)\sin\theta + (k/m)\omega \end{array}\right\} \text{ is autonomous. Dimension: 2}$$

Ex. 7: Show that given functions are solutions of initial value problem

$$\text{IVP: } \left\{\begin{array}{rcl} x' &=& -4x + 6y \\ y' &=& -3x + 5y \end{array}\right\},\ \left\{\begin{array}{l} x(0) = 0 \\ y(0) = 1 \end{array}\right\};\ \text{functions } \left\{\begin{array}{rcl} x(t) &=& 2e^{2t} - 2e^{-t} \\ y(t) &=& -e^{-t} + 2e^{2t} \end{array}\right\}$$

$$x'(t) = 4e^{2t} + 2e^{-t},\ -4x(t) + 6y(t) = -4(2e^{2t} - 2e^{-t}) + 6(-e^{-t} + 2e^{2t}) = 4e^{2t} + 2e^{-t}$$

$$y'(t) = e^{-t} + 4e^{2t},\ -3x(t) + 5y(t) = -3(2e^{2t} - 2e^{-t}) + 5(-e^{-t} + 2e^{2t}) = e^{-t} + 4e^{2t}$$

IC: $x(0) = 0,\ y(0) = 1,$

hence $x(t), y(t)$ are solutions of IVP