7.7: Determinants

For 2×2: (assume $a \neq 0$)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\xrightarrow{R1(2,1,-c/a)} \begin{bmatrix} a & b \\ 0 & d - bc/a \end{bmatrix}$$

A is nonsingular if and only if

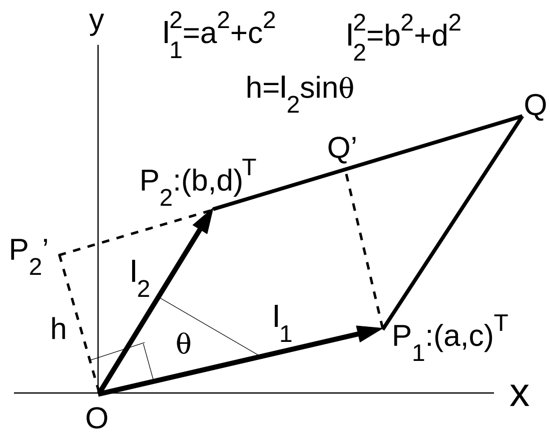
$$a(d - bc/a) = ad - bc \neq 0$$

Def.: The determinant of A is

$$\det(A) \stackrel{def}{=} ad - bc$$

Geometric Interpretation:

$|\det(A)|$ is the area of the parallelogram generated by the two column (or row) vectors of A



Area of parallelogram OP_1QP_2
$= $ area of rectangle $OP_1Q'P_2'$
$= l_1 h = l_1 l_2 \sin\theta$

dot-product:

$$[a,c]^T \cdot [b,d]^T = ab + cd = l_1 l_2 \cos\theta$$

$$\begin{aligned}
\Rightarrow l_1^2 l_2^2 \sin^2\theta &= l_1^2 l_2^2 - l_1^2 l_2^2 \cos^2\theta \\
&= (a^2+c^2)(b^2+d^2) \\
&\quad -(ab+cd)^2 \\
&= a^2d^2 + b^2c^2 - 2abcd \\
&= (ad-bc)^2 \\
\Rightarrow \text{area}(OP_1QP_2) &= |ad - bc|
\end{aligned}$$

Determinants of $\mathbf{n} \times \mathbf{n}$-Matrices

Thm.: For any $n \times n$-matrix A, there exists a unique value, det(A), called the determinant of A, such that

1. $\det(B) = \det(A)$ if $A \to B$ via $R1(i, j, \alpha)$
2. $\det(B) = -\det(A)$ if $A \to B$ via $R2(i, j)$
3. $\det(B) = c\det(A)$ if $A \to B$ via $R3(i, c)$
4. $\det(I) = 1$

Geometric Interpretation:

$|\det(A)|$ is the n-dimensional volume of the n-dimensional parallelepiped generated by the columns (or rows) of A

Properties:

- $\det(A) = 0 \Leftrightarrow A$ is singular
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = 1/\det(A)$ if A is nonsingular
- $\det(A^T) = \det(A)$
- $\det(A) = a_{11}a_{22}\cdots a_{nn}$ if A is upper triangular ($a_{ij} = 0$ for $i < j$) or lower triangular ($a_{ij} = 0$ for $i > j$)

Computation of Determinants Using Row Transformations

- Apply row operations to transform A to REF
- Keep track of all
  a: multiplications $R3(i,c)$ (factor $1/c$)
  b: row flips $R2(i,j)$ (factor -1)
- If A is singular
  $\Rightarrow$ $\det(A) = \det(REF) = 0$
- If A is nonsingular
  $\Rightarrow$ all diagonals are pivots
  $\Rightarrow$ $\det(A)$ is the product of all pivots times all factors from a and b

Ex.: $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & 1 \\ 4 & 8 & 9 \end{bmatrix} \overset{R1(3,1,-2)}{\longrightarrow}$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \overset{R2(2,3)}{\longrightarrow} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \det(A) = (-1) \cdot 2 \cdot 2 \cdot 1 = -4$

Ex.: $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 4 & -3 & 0 \end{bmatrix}$

Apply successively

$R2(1,3)$, $R1(2,1,-2)$, $R3(2,-1/2)$, $R1(3,2,-2)$, $R1(4,2,-4)$, $R3(3,1/4)$, $R1(4,3,1)$

to obtain a REF with pivot entries $1,1,1,-41/4 \Rightarrow$

$\det(A) = (-1) \cdot (-2) \cdot 4 \cdot (-41/4) = -82$

Row and Column Expansions of Determinants

$A = [a_{ij}]$: $n \times n$

Def.: The ij-minor of A is the $(n-1) \times (n-1)$-matrix A_{ij} obtained from A by deleting the i-th row and j-th column of A.

Expansion -Thm.:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \quad (1)$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \quad (2)$$

for any i or j

(1): expansion about i-th row

(2): expansion about j-th column

Ex.: $\mathbf{n = 3}$. Use (1) with $i = 1$ (vertical bars denote determinants)

$$\begin{aligned}
\det(A) &= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + (-1)^{1+3} a_{13} \det(A_{13}) \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})
\end{aligned}$$

Ex.: $A = \begin{bmatrix} -3 & 2 & 8 \\ 0 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ Use (1) with $i = 2$ (2 zeros)

$$\det(A) = (-1)^{2+2} 4 \begin{vmatrix} -3 & 8 \\ 3 & 1 \end{vmatrix} = -108$$

Determinants and Matrix Inverse

- $A = [a_{ij}]$: $n \times n$. Assume $\det(A) \neq 0 \Rightarrow A$ invertible
- Denote entries of inverse matrix by $A^{-1} = [a_{ij}^{(-1)}]$

Thm.:

$$a_{ij}^{(-1)} = \frac{(-1)^{i+j}\det(A_{ji})}{\det(A)}$$

Ex.: 2×2-Inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \det(A) = ad - bc \neq 0$$

$$A_{11} = d, \ A_{12} = c, \ A_{21} = b, \ A_{22} = a$$

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{bmatrix} A_{11} & -A_{21} \\ -A_{12} & A_{22} \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

Ex.: Solve $\begin{cases} x + 2y &= 1 \\ 3x + 4y &= 2 \end{cases}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

Ex.: $A = \begin{bmatrix} -3 & 2 & 8 \\ 0 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, find A^{-1}

$$\det(A_{11}) = \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} = 4$$

Analogously:

$$\begin{aligned} \det(A_{12}) &= 0, & \det(A_{13}) &= -12 \\ \det(A_{21}) &= -14, & \det(A_{22}) &= -27 \\ \det(A_{23}) &= -12, & \det(A_{31}) &= -32 \\ \det(A_{32}) &= 0, & \det(A_{33}) &= -12 \end{aligned}$$

$$\Rightarrow A^{-1} = \frac{1}{-108} \begin{bmatrix} 4 & 14 & -32 \\ 0 & -27 & 0 \\ -12 & 12 & -12 \end{bmatrix}$$

Worked Out Examples from Exercises

Ex. 11: Compute determinant of given matrix using row transformations

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 2 \end{bmatrix} \overset{R2(1,3)}{\longrightarrow} \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & -1 & 3 & 4 \\ -1 & 3 & 1 & 2 \end{bmatrix} \overset{R1(3,1,2),R1(4,1,-1)}{\longrightarrow}$$

$$\begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 3 & 3 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix} \overset{R3(2,1/2)}{\longrightarrow} \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 3 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix} \overset{R1(3,2,-3),R1(4,2,-1)}{\longrightarrow}$$

$$\begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix} \overset{R2(3,4)}{\longrightarrow} \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 6 & 4 \end{bmatrix} \overset{R1(4,3,-3)}{\longrightarrow} \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\Rightarrow \det(A) = (-1)\cdot 2 \cdot (-1) \cdot [(-1)\cdot 1 \cdot 2 \cdot (-2)] = 8$$

The first two (-1)'s are due to $R2(1,3)$ and $R2(3,4)$. The first 2 is due to $R3(2,1/2)$.

Ex. 20: Compute determinant of matrix A from Ex. 11 using row or column expansions

Expand about 2nd row:

$$\begin{aligned}
\begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 2 \end{vmatrix} &= (-1)^{2+2}2\begin{vmatrix} 2 & 3 & 4 \\ -1 & 0 & 0 \\ -1 & 1 & 2 \end{vmatrix} + (-1)^{2+3}(-2)\begin{vmatrix} 2 & -1 & 4 \\ -1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} \\
&= \quad 2(-1)^{2+1}(-1)\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} \\
&\quad +2[(-1)^{2+1}(-1)\begin{vmatrix} -1 & 4 \\ 3 & 2 \end{vmatrix} + (-1)^{2+2}2\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}] \\
&= \quad 2(3 \cdot 2 - 4 \cdot 1) \\
&\quad +2[((-1) \cdot 2 - 3 \cdot 4) + 2(2 \cdot 2 - (-1) \cdot 4)] \\
&= 4 + 2(-14 + 16) \\
&= 8
\end{aligned}$$

Ex. 26: Compute determinant and decide if columns of A are linearly dependent. If yes, find basis of nullspace.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Apply $R(1,3) \Rightarrow$ lower triangular matrix with 1's in the diagonal
$\Rightarrow \det(A) = (-1) \cdot 1 = -1 \Rightarrow$ linearly independent

Ex. 27: Same as Ex. 26 for $A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$

Expand about 3rd row:

$$\det(A) = (-1)^{3+1} 3 \begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix} = 3[(-2) \cdot 2 - (-4) \cdot 1] = 0$$

$\Rightarrow$ columns are linearly dependent. Find basis of nullspace:

$$A \overset{R2(1,3),R3(1,1/3)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & -2 & -4 \end{bmatrix} \overset{R1(2,1,-2),R1(3,1,-1)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\overset{R1(3,2,2)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Set free variable $x_3 = 1 \;\Rightarrow\; x_1 = 0$, $x_2 = -2$

$\Rightarrow$ basis $[0, -2, 1]^T$

Ex. 36: For which values of x has A a nontrivial nullspace?

$$A = \begin{bmatrix} -1-x & 5 & 2 \\ 0 & -x & -1 \\ 0 & 6 & -5-x \end{bmatrix}.$$ Expand determinant about first column:

$$\begin{aligned} \det(A) &= (-1)^{1+1}(-1-x)\begin{vmatrix} -x & -1 \\ 6 & -5-x \end{vmatrix} = (-1-x)[(-x)(-5-x)+6] \\ &= -(1+x)(x^2+5x+6) = -(1+x)(x+2)(x+3) \end{aligned}$$

$$\det(A) = 0 \text{ for } x = -1, -2, -3$$

$\Rightarrow$ nontrivial nullspace for these values of x

Ex. 49: Compute determinant of A; decide if there is a nontrivial nullspace

$$A = \begin{bmatrix} 556 & 65 & -91 & 52 & 416 & -143 \\ 550 & 60 & -90 & 50 & 410 & -140 \\ -169 & -22 & 26 & -18 & -131 & 38 \\ -96 & -13 & 14 & -7 & -69 & 27 \\ 550 & 60 & -90 & 50 & 410 & -140 \\ 825 & 97 & -137 & 80 & 617 & -211 \end{bmatrix}$$

After editing A in Matlab, the determinant is computed using the command *det(A)*. Matlab output: $\det(A) = 0 \Rightarrow A$ has a nontrivial nullspace.