7.6 Square Matrices

A: square matrix ($\mathbf{n} \times \mathbf{n}$)

Def.: A is nonsingular if for any $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has a solution

Thm.: The following properties are equivalent:

- A is nonsingular
- $REF(A)$ has no free variables
- $RREF(A) = I$ (identity matrix)
- $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

Def.: A is invertible if there exists a unique matrix B s.t.

$$AB = BA = I$$

Set $B = A^{-1}$ (inverse matrix)

Thm.:

(a) A is invertible $\Leftrightarrow$ A is nonsingular

(b) If A is invertible, the unique solution of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = A^{-1}\mathbf{b}$$

(c) If A is invertible, then

$$RREF([A, I]) = [I, A^{-1}]$$

Use this to compute A^{-1}

Ex. $A = \begin{bmatrix} -3 & 6 & 8 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Matlab $\Rightarrow RREF(A) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

x_2 = free variable $\Rightarrow$ A is singular

(a) Consider $A\mathbf{x} = \mathbf{b}$ for

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Augmented matrix:

$$M = [A, \mathbf{b}] = \begin{bmatrix} -3 & 6 & 8 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Matlab $\Rightarrow$

$$RREF(M) = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last row requires

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$\Rightarrow$ no solution

(b) Consider $A\mathbf{x} = \mathbf{b}$ for

$$\mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \Rightarrow M = \begin{bmatrix} -3 & 6 & 8 & 5 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Matlab $\Rightarrow$

$$RREF(M) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- x_1, x_3: pivot variables
- $x_2 = t$: free variable

Equations: $\begin{cases} x_1 = 1 + 2t \\ x_3 = 1 \end{cases}$

$\Rightarrow \infty$ many solutions:

$$\mathbf{x} = \begin{bmatrix} 1 + 2t \\ t \\ 1 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Ex. $A = \begin{bmatrix} 3 & -4 & -8 \\ 2 & -3 & -10 \\ 0 & 0 & 2 \end{bmatrix} \overset{R1(2,1,-2/3)}{\longrightarrow} \begin{bmatrix} -3 & -4 & -8 \\ 0 & -1/3 & -14/3 \\ 0 & 0 & 2 \end{bmatrix} = REF(A)$

All columns pivots $\Rightarrow$ A nonsingular. Matlab $\Rightarrow$ $RREF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Consider $A\mathbf{x} = \mathbf{b}$ for

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Augmented matrix:

$$M = [A, \mathbf{b}] = \begin{bmatrix} 3 & -4 & -8 & 0 \\ 2 & -3 & -10 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Matlab $\Rightarrow$

$$RREF(M) = \begin{bmatrix} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$\Rightarrow$ unique solution:

$$\mathbf{x} = \begin{bmatrix} -12 \\ -10 \\ 1/2 \end{bmatrix}$$

(b) Find A^{-1} $\rightarrow$ augment A by I:

$$M = [A, I] = \begin{bmatrix} 3 & -4 & -8 & 1 & 0 & 0 \\ 2 & -3 & -10 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

Matlab $\Rightarrow$

$$RREF(M) = \begin{bmatrix} 1 & 0 & 0 & 3 & -4 & -8 \\ 0 & 1 & 0 & 2 & -3 & -7 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\Rightarrow \; A^{-1} = \begin{bmatrix} 3 & -4 & -8 \\ 2 & -3 & -7 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Compute: $A^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -10 \\ 1/2 \end{bmatrix}$

Worked Out Examples

(A) Find all solutions to $A\mathbf{x} = \mathbf{0}$. Is A singular?

Ex. 12: $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \overset{R1(2,1,-1)}{\longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = REF(A).$

No free variables $\Rightarrow$ A non-singular $\Rightarrow$ only solution is $\mathbf{x} = \mathbf{0}$

Ex.: $A = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \overset{R1(2,1,3)}{\longrightarrow} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = REF(A).$

Free variable: $x_2 = t$, equation: $x_1 - t = 0 \Rightarrow$ solutions: $\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

System has nontrivial solutions $\Rightarrow$ A singular.

Ex. 14: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \overset{R1(2,1,-1),R1(3,1,-1)}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$$\overset{R2(2,3)}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = REF(A)$$

All columns pivot $\Rightarrow$ only trivial solution $\mathbf{x} = \mathbf{0} \Rightarrow A$ nonsingular

(B) Which matrices are singular? If A is nonsingular find A^{-1}

Ex.: $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ is REF with no free variables $\Rightarrow$ A nonsingular

$$M = [A, I] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} \overset{R2(2,1/4)}{\longrightarrow} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1/4 \end{bmatrix}$$

$$\overset{R1(1,2,-2)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & 1/4 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/4 \end{bmatrix}$$

Ex. 23: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is REF with no free variables $\Rightarrow$ A nonsingular

$$M = [A, I] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \overset{R1(1,3,-1),R1(2,3,-1)}{\longrightarrow} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R1(1,2,-1)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(C) Without solving find which systems have unique solutions

Ex. 28: $\left.\begin{array}{rcl} x_1 + 2x_2 & = & 4 \\ x_1 - x_2 & = & 6 \end{array}\right\} \Rightarrow A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \overset{R1(2,1,-1)}{\longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = REF(A)$

$REF(A)$ has no free variables $\Rightarrow$ A nonsingular $\Rightarrow$ unique solution

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Ex. 33: $A\mathrm{x} = \mathrm{b}$ for $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$, $\mathrm{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} \overset{R1(2,1,1)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \overset{R1(3,2,-2)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = REF(A)$$

$REF(A)$ has free variable $\Rightarrow$ A singular $\Rightarrow$ not unique solutions

Ex.: Given $A = \begin{bmatrix} 0 & 2 & -4 \\ 3 & -5 & 10 \\ 2 & -4 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$,

for what values of a, b does $A\mathbf{x} = \mathbf{b}$ have solutions?

$$M = [A, \mathbf{b}] = \begin{bmatrix} 0 & 2 & -4 & 0 \\ 3 & -5 & 10 & a \\ 2 & -4 & 8 & b \end{bmatrix} \overset{R2(1,3)}{\longrightarrow} \begin{bmatrix} 2 & -4 & 8 & b \\ 3 & -5 & 10 & a \\ 0 & 2 & -4 & 0 \end{bmatrix} \overset{R1(2,1,-3/2)}{\longrightarrow}$$

$$\begin{bmatrix} 2 & -4 & 8 & b \\ 0 & 1 & -2 & a - 3b/2 \\ 0 & 2 & -4 & 0 \end{bmatrix} \overset{R1(3,2,-2)}{\longrightarrow} \begin{bmatrix} 2 & -4 & 8 & b \\ 0 & 1 & -2 & a - 3b/2 \\ 0 & 0 & 0 & 3b - 2a \end{bmatrix} = REF(A)$$

system has solutions if 4th column in $REF(A)$ is *not* pivot,

hence if $3b - 2a = 0$