7.5 Span of a Set of Vectors

Def.: Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathbf{R}^n$, the set of all their linear combinations is called their *span*, and is denoted by

$$\text{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

Thm.: $\text{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is a subspace of $\mathbf{R}^n$

Def.: If V is a subspace of $\mathbf{R}^n$, the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathbf{R}^n$ are called a spanning set for V if

$$V = \text{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

$\Rightarrow$ any $\mathbf{x} \in V$ can be written as

$$\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \ldots + t_k\mathbf{x}_k$$

with numbers $t_1, \ldots, t_k$

Thm.: Every subspace has a spanning set

Nullspaces:

Ex.: Consider $A\mathbf{x} = \mathbf{0}$ for

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Free variable: $x_3 = t$

Equations: $-x_2 = 0$, $x_1 - 2t = 0$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{null}(A) = \text{span}([2, 0, 1]^T)$$

Ex.: $A\mathbf{x} = \mathbf{0}$ for $A = [1, 3, -2]$

Free variables: $x_2 = s$, $x_3 = t$

Equation: $x_1 + 3s - 2t = 0$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} 2t - 3s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{null}(A) = \text{span}([-3, 1, 0]^T, [2, 0, 1]^T)$$

Determine if a given $\mathbf{x}$ is in span$(\mathbf{x}_1, \ldots, \mathbf{x}_k)$:

1. Form matrix
$$X = [\mathbf{x}_1, \ldots, \mathbf{x}_k]$$
2. Try to solve the system
$$X\mathbf{c} = \mathbf{x}$$
for $\mathbf{c}$
3. If $X\mathbf{c} = \mathbf{x}$ has no solution (system inconsistent), $\mathbf{x}$ is not in span$(\mathbf{x}_1, \ldots, \mathbf{x}_k)$
4. If $X\mathbf{c} = \mathbf{x}$ has a solution $\mathbf{c} = [c_1, \ldots, c_k]^T$, then
$$\mathbf{x} = c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k$$
is in span$(\mathbf{x}_1, \ldots, \mathbf{x}_k)$

Ex.: $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$$\Rightarrow X = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

augmented matrix for $X\mathbf{c} = \mathbf{x}$:

$$M = [X, \mathbf{x}]$$

(a) Let $\mathbf{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$:

$$M = \begin{bmatrix} -1 & 1 & 5 \\ 2 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

$\Rightarrow$ solution exists, $c_1 = -2, c_2 = 3$
$\Rightarrow$ $\mathbf{x} = -2\mathbf{x}_1 + 3\mathbf{x}_2$ is in span$(\mathbf{x}_1, \mathbf{x}_2)$

(b) Let $\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$ be arbitrary:

$$M = \begin{bmatrix} -1 & 1 & u \\ 2 & 1 & v \end{bmatrix} \to \begin{bmatrix} 1 & 0 & (v-u)/3 \\ 0 & 1 & (2u+v)/3 \end{bmatrix}$$

Solution exists $\Rightarrow$ span$(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{R}^2$

Ex.: $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$,

$$\Rightarrow X = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Last column pivot
$\Rightarrow$ solutions don't exist
$\Rightarrow$ $\mathbf{x}$ is not in span$(\mathbf{x}_1, \mathbf{x}_2)$

Note: $\mathbf{x}_2 = 2\mathbf{x}_1$

$$\begin{aligned} \Rightarrow\ c_1\mathbf{x}_1 + c_2\mathbf{x}_2 &= (c_1 + 2c_2)\mathbf{x}_1 \\ &= (c_1/2 + c_2)\mathbf{x}_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow\ \text{span}(\mathbf{x}_1, \mathbf{x}_2) &= \text{span}(\mathbf{x}_1) \\ &= \text{span}(\mathbf{x}_2) \end{aligned}$$

Ex.: $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

General vector in span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$$

Since $\mathbf{x}_3 = \mathbf{x}_2 - \mathbf{x}_1 \Rightarrow$

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3(\mathbf{x}_2 - \mathbf{x}_1) \\ &= (c_1 - c_3)\mathbf{x}_1 + (c_2 + c_3)\mathbf{x}_2 \end{aligned}$$

$$\Rightarrow\ \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \text{span}(\mathbf{x}_1, \mathbf{x}_2)$$

and on p.2 it was shown that

$$\text{span}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{R}^2$$

Linear Dependence and Independence

Def.: $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbf{R}^n$ are

- linearly independent if the only linear combination of them that is $\mathbf{0}$ is trivial, i.e.

$$c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k = \mathbf{0} \quad (1)$$
$$\Rightarrow \; c_1 = c_2 = \cdots = c_k = 0$$

- linearly dependent if there are numbers $c_1, \ldots, c_k$, not all zero, for which (1) is satisfied.

Linear independence check

$$(1) \;\Rightarrow\; X\mathbf{c} = \mathbf{0} \qquad (2)$$

Thm.: $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are

- linearly independent if (2) has only $\mathbf{c} = \mathbf{0}$ as solution
- linearly dependent if (2) has nontrivial solutions

If $k > n$, $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are always linearly dependent

Ex.: $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \overset{R1(2,1,-1)}{\longrightarrow} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$\Rightarrow$ $X\mathbf{c} = \mathbf{0}$ has only solution $\mathbf{c} = \mathbf{0}$
$\Rightarrow$ $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent

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Ex.: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ the 3 vectors are linearly dependent ($k = 3 > 2$)

Ex.: $\mathbf{x}_j = \text{col}_j(X)$, $j = 1, 2, 3$, where

$$X = \begin{bmatrix} 0 & -2 & -2 \\ -2 & -1 & -3 \\ 2 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$X\mathbf{c} = \mathbf{0}$ for $\mathbf{c} = [1, 1, -1]^T \Rightarrow$

$\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0} \Rightarrow$ linearly dependent

Bases and Dimension of a Subspace, Rank of a Matrix

Def.: A spanning set $\mathbf{x}_1, \ldots, \mathbf{x}_k$ for a subspace V of $\mathbf{R}^n$ is a basis of V if $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent.

Meaning:

$\mathbf{x} \in V \Rightarrow \mathbf{x} = a_1\mathbf{x}_1 + \ldots + a_k\mathbf{x}_k$ with *unique* numbers $a_1, \ldots, a_k$

Thm.:

1. Every subspace V has a basis (in fact, ∞ many)
2. All bases of V have the same number of vectors

Def.: The dimension of a subspace V of $\mathbf{R}^n$ is the number of vectors in a basis of V, and denoted by $\dim V$.

Def.: The rank of a matrix X is the number of pivots in an REF of X, and denoted by $\operatorname{rank} X$.

Thm.: Given a spanning set $\mathbf{x}_1, \ldots, \mathbf{x}_k$ for a subspace V of $\mathbf{R}^n$, let $X = [\mathbf{x}_1, \ldots, \mathbf{x}_k]$. Then

1. $\dim V = \operatorname{rank} X$
2. $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is a basis of V if and only if $\operatorname{rank} X = k$
3. If $k = n$ and $\operatorname{rank} X = n$, then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ form a basis of $\mathbf{R}^n$ $\quad(\dim \mathbf{R}^n = n)$

Ex.: Let $\mathbf{e}_j = \text{col}_j(I)$
where I: $n \times n$ identity matrix

$\mathbf{e}_1, \ldots, \mathbf{e}_n$ are a basis of $\mathbf{R}^n$
– called the standard basis

For $n = 2$: $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Ex.: $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Claim: $\mathbf{x}_1, \mathbf{x}_2$ are a basis of $\mathbf{R}^2$

Proof: Given $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, show that $\mathbf{x}$ can be uniquely represented as

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$$

Equations for a_1, a_2:

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \end{aligned}$$

$$X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \stackrel{R1(2,1,-1)}{\longrightarrow} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

is nonsingular $\Rightarrow$ unique solution

Ex.: $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

null$(A) = \text{span}([2, 0, 1]^T)$ (see p.1)

$[2, 0, 1]^T$ is a basis of null(A)

$\Rightarrow$ dim null$(A) = 1$

Ex.: $A = [1, 3, -2]$ (see p.1)

null$(A) = \text{span}([-3, 1, 0]^T, [2, 0, 1]^T)$

$[-3, 1, 0]^T, [2, 0, 1]^T$
are linearly independent

$\Rightarrow$ $[-3, 1, 0]^T, [2, 0, 1]^T$
are a basis of null(A)

$\Rightarrow$ dim null$(A) = 2$

Computation of a Basis of a Nullspace

A: $m \times n$

- Transform $A \to REF(A)$ or $RREF(A)$
- For each choice of a free variable set this variable equal to 1 and all other free variables equal to 0
- For each of these choices solve for the pivot variables
- $\Rightarrow f$ ($= \sharp$ of free variables) solution vectors $\mathbf{x}_1, \ldots, \mathbf{x}_f$ for $A\mathbf{x} = \mathbf{0}$
- $\mathbf{x}_1, \ldots, \mathbf{x}_f$ are a basis of null(A)

Ex.: $A = \begin{bmatrix} 3 & 1 & 1 & -2 \\ -6 & 1 & -2 & 4 \\ 12 & 1 & 4 & -8 \\ 6 & 2 & 2 & -4 \end{bmatrix}$

Matlab $\Rightarrow$

$$RREF(A) = \begin{bmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables: x_3, x_4; and $x_2 = 0$

(1) Set $x_3 = 1$, $x_4 = 0 \Rightarrow x_1 = -1/3$

$$\Rightarrow \mathbf{x}_1 = [-1/3, 0, 1, 0]^T$$

(2) Set $x_3 = 0$, $x_4 = 1 \Rightarrow x_1 = 2/3$

$$\Rightarrow \mathbf{x}_2 = [2/3, 0, 0, 1]^T$$

$\mathbf{x}_1, \mathbf{x}_2$ are a basis of null(A)

dim null$(A) = 2$

Solutions of Inhomogeneous Systems and Nullspaces

Form of general solution to $A\mathbf{x} = \mathbf{b}$:

$$\mathbf{x} = \mathbf{x}_p + t_1\mathbf{x}_1 + \ldots + t_f\mathbf{x}_f$$

where

- $\mathbf{x}_p$: particular solution
- $\mathbf{x}_1, \ldots, \mathbf{x}_f$: basis of null($A$)
- $t_1, \ldots, t_f$: free parameters

Finding $\mathbf{x}_p$**:**

- Transform $M = [A, \mathbf{b}]$ to $REF(M)$ or $RREF(M)$
- Set all free variables 0 and solve for pivot variables

Ex.: $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 4 & -2 \\ 2 & 3 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix}$$

Augmented matrix: $M = [A, \mathbf{b}]$.
Matlab $\Rightarrow$

$$RREF(M) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variable: x_3

Set $x_3 = 0 \Rightarrow \begin{cases} x_1 & = & 1 \\ x_2 & = & -2 \end{cases}$

$\Rightarrow \mathbf{x}_p = [1, -2, 0]^T$

$$RREF(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \mathbf{x}_1 = [-1, 1, 1]^T$ is basis of null(A)

Solution set: $\{\mathbf{x} = \mathbf{x}_p + t\mathbf{x}_1 \mid t \in \mathbf{R}\}$

Worked Out Examples

(A) Is $\mathbf{w}$ in the span of the given vectors? If yes, find linear combination of spanning vectors for $\mathbf{w}$.

Ex. 1: $\mathbf{u}_1 = [1, -2]^T$, $\mathbf{u}_2 = [3, 0]^T$. Is $\mathbf{w} = [5, -2]^T$ in span$(\mathbf{u}_1, \mathbf{u}_2)$?

Set $U = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$; $U\mathbf{c} = \mathbf{w} \to M = [U, \mathbf{w}] = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 0 & -2 \end{bmatrix}$

$$M \to \begin{bmatrix} 1 & 3 & 5 \\ 0 & 6 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4/3 \end{bmatrix} \Rightarrow \left\{ \begin{array}{c} \text{yes, } \mathbf{c} = [1, 4/3]^T \\ \mathbf{w} = \mathbf{u}_1 + (4/3)\mathbf{u}_2 \end{array} \right\}$$

Ex. 3: $\mathbf{u}_1 = [1, -2]^T$, $\mathbf{u}_3 = [2, -4]^T$. Is $\mathbf{w} = [3, -3]^T$ in span$(\mathbf{u}_1, \mathbf{u}_3)$?

Here $M = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow$ inconsistent

$\Rightarrow$ no, $\mathbf{w}$ is not in span$(\mathbf{u}_1, \mathbf{u}_3) =$ span$(\mathbf{u}_1) =$ span$(\mathbf{u}_3)$

Ex. 7: $\mathbf{v}_1 = [1, -4, 4]^T$, $\mathbf{v}_2 = [0, -2, 1]^T$, $\mathbf{v}_3 = [1, -2, 3]^T$.
Is $\mathbf{w} = [1, 0, 2]^T$ in span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -4 & -2 & -2 & 0 \\ 4 & 1 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 1 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ 1-parameter family of solutions. Choose, e.g., $c_3 = 0 \Rightarrow c_1 = 1$, $c_2 = 2$

$\Rightarrow$ yes, $\mathbf{w} = \mathbf{v}_1 - 2\mathbf{v}_2 + 0\mathbf{v}_3$ is in span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

(B) Either show that the given vectors are linearly independent or find nontrivial linear combination that adds to zero

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Ex. 17: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$; $X = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \overset{R1(2,1,-2)}{\longrightarrow} \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} (REF)$

REF has no free variables $\Rightarrow$ linearly independent

Ex. 20: $\mathbf{v}_1 = \begin{bmatrix} -8 \\ 9 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 7 \end{bmatrix}$; $X = \begin{bmatrix} -8 & -2 \\ 9 & 0 \\ -6 & 7 \end{bmatrix} \overset{R3(1,-1/8)}{\longrightarrow} \begin{bmatrix} 1 & 1/4 \\ 9 & 0 \\ -6 & 7 \end{bmatrix}$

$$\overset{R1(2,1,-9),R1(3,1,6)}{\longrightarrow} \begin{bmatrix} 1 & 1/4 \\ 0 & -9/4 \\ 0 & 17/2 \end{bmatrix} \overset{R1(3,2,34/9)}{\longrightarrow} \begin{bmatrix} 1 & 1/4 \\ 0 & -9/4 \\ 0 & 0 \end{bmatrix} (REF)$$

REF has no free variables $\Rightarrow$ linearly independent

Ex. 22: $\mathbf{v}_1 = \begin{bmatrix} -8 \\ 9 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 8 \\ -18 \\ 40 \end{bmatrix}$; $X = \begin{bmatrix} -8 & -2 & 8 \\ 9 & 0 & -18 \\ -6 & 7 & 40 \end{bmatrix}$

$X \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} (RREF)$ free variable: c_3, set $c_3 = 1 \Rightarrow c_1 = 2$, $c_2 = -4$

$\Rightarrow 2\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$

(C) Determine if nullspace of matrix is trivial ($\text{null}(A) = \mathbf{0}$) or nontrivial. If nontrivial, find a basis.

Ex. 25: $A = [2, -1]$ (REF), free variable: y

set $y = 1 \Rightarrow 2x - 1 = 0 \Rightarrow x = 1/2 \Rightarrow$ basis $[1/2, 1]^T$

Ex. 28: $A = \begin{bmatrix} 4 & 4 \\ -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{null}(A) = \mathbf{0}$

Ex.: $A = \begin{bmatrix} 0 & -2 & 0 & -2 \\ 2 & -12 & -4 & -14 \\ 0 & 1 & 0 & 1 \\ -2 & 11 & 4 & 13 \end{bmatrix} \rightarrow \ldots \rightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $(RREF)$

free variables: x_3, x_4

set $x_3 = 1$, $x_4 = 0 \Rightarrow x_1 - 2 = 0$, $x_2 = 0 \Rightarrow \mathbf{x}_1 = [2, 0, 1, 0]^T$

set $x_3 = 0$, $x_4 = 1 \Rightarrow x_1 - 1 = 0$, $x_2 + 1 = 0 \Rightarrow \mathbf{x}_2 = [1, -1, 0, 1]^T$

$\mathbf{x}_1, \mathbf{x}_2$ are a basis of $\text{null}(A)$

(D) Find solution set of $A\mathbf{x} = \mathbf{b}$ using previously computed basis of null(A).

Ex.: A as in Ex. 25, $\mathbf{b} = 2$.

$M = [A, \mathbf{b}] = [2, -1, 2]\,(REF)$, free variable: y, set $y = 0$

$\Rightarrow 2x = 2 \Rightarrow x = 1 \Rightarrow$ particular solution: $\mathbf{x}_p = [1, 0]^T$

Use basis of nullspace from Ex. 25

$\Rightarrow$ solution set $\{\mathbf{x} = [1, 0]^T + t[1/2, 1]^T \,|\, t \in \mathbf{R}\}$

Ex.: A as in Ex. 28, $\mathbf{b} = [0, -1]^T$.

$$[A, \mathbf{b}] = \begin{bmatrix} 4 & 4 & 0 \\ -2 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Equations: $x = 1$, $y = -1 \Rightarrow$ unique solution $\mathbf{x} = [1, -1]^T$

Ex.: A as in last Ex. of (C), p.11; $\mathbf{b} = [0, 6, 0, -6]^T$.

$$[A, \mathbf{b}] = \begin{bmatrix} 0 & -2 & 0 & -2 & 0 \\ 2 & -12 & -4 & -14 & 6 \\ 0 & 1 & 0 & 1 & 0 \\ -2 & 11 & 4 & 13 & -6 \end{bmatrix} \rightarrow \ldots \rightarrow \begin{bmatrix} 1 & 0 & -2 & -1 & 3 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (RREF)$$

set free variables $x_3 = x_4 = 0 \Rightarrow x_1 = 3$, $x_2 = 0$

$\Rightarrow$ particular solution $\mathbf{x}_p = [3, 0, 0, 0]^T$. Use basis of nullspace from Ex. on p.11

$\Rightarrow$ solution set $\{\mathbf{x} = [3, 0, 0, 0]^T + s[2, 0, 1, 0]^T + t[1, -1, 0, 1]^T \,|\, s, t \in \mathbf{R}\}$