7.3: Solving Systems of Equations

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

Matrix-vector notation:

$$A\mathbf{x} = \mathbf{b}$$

$A = [a_{ij}]_{mn}$: $m \times n$ matrix

$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$: target vector

$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$: unknown vector

- length of $\mathbf{x}$ = row–length of A
- length of $\mathbf{b}$ = column–length of A

Examples

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 5 \\ 4x_1 & - & x_2 & = & 0 \end{array}$$

or

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\begin{array}{rcrcrcr} x_1 & - & 4x_2 & + & x_3 & = & -2 \\ -2x_1 & + & 10x_2 & - & 3x_3 & = & 4 \end{array}$$

or

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -4 & 1 \\ -2 & 10 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Solution Procedure: Example

$$\begin{aligned} x + 2y &= 5 \quad (1) \\ 4x - y &= 0 \quad (2) \end{aligned}$$

Solve by Substitution:

Solve (1) for x

$$\Rightarrow \; x = 5 - 2y$$

Sub x in (2)

$$\Rightarrow \; 4(5-2y) - y = 20 - 9y = 0$$

$$\Rightarrow \; y = 20/9$$

Hence:

$$\begin{aligned} x + 2y &= 5 \quad (3) \\ y &= 20/9 \quad (4) \end{aligned}$$

Subtract 2 × (4) from (3)

$$\begin{aligned} \Rightarrow \; x &= 5/9 \\ y &= 20/9 \end{aligned}$$

Matrix Approach:

$$A\mathbf{x} = \mathbf{b} : \;\; \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \; A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

- Form augmented matrix:

$$M = [A, \mathbf{b}] = \begin{bmatrix} 1 & 2 & 5 \\ 4 & -1 & 0 \end{bmatrix}$$

- Subtract 4 × 1st row of M from 2nd row

$$\Rightarrow \; M_1 = [A, \mathbf{b}] = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -9 & -20 \end{bmatrix}$$

- Multiply 2nd row of M_1 by $-1/9$

$$\Rightarrow \; M_2 = [A, \mathbf{b}] = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 20/9 \end{bmatrix}$$

- Subtract 2 × 2nd row of M_2 from 1st row

$$\Rightarrow \; M_3 = [A, \mathbf{b}] = \begin{bmatrix} 1 & 0 & 5/9 \\ 0 & 1 & 20/9 \end{bmatrix}$$

- Read off solution from M_3:
  $x = 5/9$, $y = 20/9$

Row Operations and Row Echelon Forms

Allowed Row Operations on a matrix M:

$R1 = R1(i, j, a)$: $\text{row}_i(M) \to \text{row}_i(M) + a\,\text{row}_j(M)$

$R2 = R2(i, j)$: Interchange rows i and j

$R3 = R3(i, a)$: $\text{row}_i(M) \to a\,\text{row}_i(M)$

(A) Apply row operations to M top left $\to$ bottom right

$\Rightarrow$ **Row Echelon Form** (not unique)

$$\begin{bmatrix} P & * & * & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- P: nonzero numbers
- $*$: arbitrary numbers
- column j with a P: pivot column
- variable x_j: pivot variable
- other variables: non–pivot variables

(B) Apply row operations to row echelon form

proceed right $\rightarrow$ left

$\Rightarrow$ Unique **Row Reduced Echelon Form** (RREF)

$$\begin{bmatrix} 1 & 0 & * & * & 0 & 0 & * & * & 0 & * \\ 0 & 1 & * & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Properties of RREF:

- Each pivot number P is 1
- All entries in pivot columns are 0 except pivot entry

Gaussian Elimination Procedure

Task: Find all solutions to

$$A\mathbf{x} = \mathbf{b} \qquad (5)$$

Procedure:

- Form **augmented matrix**
  $$M = [A, \mathbf{b}]$$
- **Step (A):** apply row operations to M to find a row echelon form REF
- **Solvability Criterion:** Last column of REF is <u>not</u> a pivot column
- **If (5) is solvable:**

- **Step (B):** Compute RREF $\rightarrow$ can read off solutions
- **or:** solve equations for REF via back-substitution

Ex.: $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

Augmented matrix:

$$M = \begin{bmatrix} 1 & 2 & 5 \\ 4 & -1 & 0 \end{bmatrix}$$

Step (A): Apply $R1(2, 1, -4)$ to M

$$\Rightarrow M_1 = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -9 & -20 \end{bmatrix} \quad \text{(REF)}$$

3rd column not pivot $\Rightarrow$ solvable

Step (B): Apply $R3(2, -1/9)$ to M_1

$$\Rightarrow M_2 = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 20/9 \end{bmatrix}$$

Apply $R1(1, 2, -2)$ to M_2

$$\Rightarrow M_3 = \begin{bmatrix} 1 & 0 & 5/9 \\ 0 & 1 & 20/9 \end{bmatrix} \quad \text{(RREF)}$$

Read off solution: $x = 5/9$, $y = 20/9$

Families of Solutions

- If REF has non-pivot columns, associate to each non-pivot variable a free parameter $\Rightarrow$
- number of free parameters $=$ number of non-pivot columns

Ex. for non-solvable equation

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Augmented matrix: $M = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \end{bmatrix}$

step (A): apply $R1(2,1,-2)$ to M

$$\Rightarrow M_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \text{ (REF)}$$

Last column is pivot $\Rightarrow$ no solution

Ex.: $\begin{matrix} x & + & y & = & 2 \\ 2x & + & 2y & = & 4 \end{matrix}$ or

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{b}; \;\; A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Augmented matrix:

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

step (A): Apply $R(2,1,-2)$ to $M \Rightarrow$

$$M_1 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ (RREF)}$$

- **pivot:** 1st column $\Rightarrow$ pivot variable: x
- **non-pivot:** 2nd column $\Rightarrow$ free parameter $y = t$
- **equation:** $x + y = 2 \Rightarrow$ $x = 2 - t$

Vector Form of Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Ex.:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 3 \\ x_1 + 2x_2 + 3x_3 + x_4 &= 6 \\ 2x_1 + 3x_2 + 4x_3 + 2x_4 &= 9 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \end{bmatrix}$$

Augmented matrix:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 1 & 6 \\ 2 & 3 & 4 & 2 & 9 \end{bmatrix}$$

Apply $R1(2,1,-1)$ and $R1(3,1,-2)$

$$\Rightarrow \ M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 1 & 2 & 0 & 3 \end{bmatrix}$$

Apply $R1(3,2,-1)$ to M_1

$$\Rightarrow \ M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(REF)}$$

Last column is not pivot $\Rightarrow$ solutions exist

Apply $R1(1,2,-1)$ to M_2

$$\Rightarrow \ M_3 = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(RREF)}$$

Non-pivot variables:

$$x_3 = s, \ x_4 = t$$

Pivot variables: x_1, x_2.

Equations for x_1, x_2 from RREF:

$$\left.\begin{aligned} x_1 - x_3 + x_4 &= 0 \\ x_2 + 2x_3 &= 3 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x_1 &= s - t \\ x_2 &= 3 - 2s \end{aligned}\right.$$

Vector form of solutions:

$$\mathbf{x} = \begin{bmatrix} s - t \\ 3 - 2s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Ex.:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 + 7x_5 &= 8 \\ 2x_1 + 4x_2 + 6x_3 + 9x_4 + 15x_5 &= 2 \\ x_2 + 3x_3 + 2x_4 + 2x_5 &= 1 \end{aligned}$$

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]^T$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 2 & 4 & 6 & 9 & 15 \\ 0 & 1 & 3 & 2 & 2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$$

Augmented matrix:

$$M = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 & 8 \\ 2 & 4 & 6 & 9 & 15 & 2 \\ 0 & 1 & 3 & 2 & 2 & 1 \end{bmatrix}$$

Apply $R1(2, 1, -2)$

$$\Rightarrow \ M_1 = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 & 8 \\ 0 & 0 & 0 & -1 & 1 & -14 \\ 0 & 1 & 3 & 2 & 2 & 1 \end{bmatrix}$$

Apply $R2(2, 3)$ (flip rows 2 and 3):

$$\Rightarrow M_2 = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 & 8 \\ 0 & 1 & 3 & 2 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & -14 \end{bmatrix} \text{REF}$$

Free variables: $x_3 = s$, $x_5 = t$

Solve for x_4, x_2, x_1
via back-substitution using REF:

3rd row: $-x_4 + x_5 = -14 \Rightarrow x_4 = 14 + t$

2nd row: $x_2 + 3x_3 + 2x_4 + 2x_5 = 1$

$$\begin{aligned} \Rightarrow x_2 &= 1 - 3x_3 - 2x_4 - 2x_5 \\ &= 1 - 3s - 2(14 + t) - 2t \\ &= -27 - 3s - 4t \end{aligned}$$

1st row: $x_1 + 2x_2 + 3x_3 + 5x_4 + 7x_5 = 8$

$$\begin{aligned} \Rightarrow x_1 &= 8 - 2x_2 - 3x_3 - 5x_4 - 7x_5 \\ &= 8 - 2(-27 - 3s - 4t) \\ &\quad -3s - 5(14 + t) - 7t \\ &= -8 + 3s - 4t \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} 3s - 4t - 8 \\ -3s - 4t - 27 \\ s \\ t + 14 \\ t \end{bmatrix} = s\mathbf{u} + t\mathbf{v} + \mathbf{w}$$

$\mathbf{u} = [3, -3, 1, 0, 0]^T$, $\mathbf{v} = [-4, -4, 0, 1, 1]^T$

$\mathbf{w} = [-8, -27, 0, 14, 0]^T$

Find same result via RREF

Worked out Examples

Ex. 1: $-2x + 4y = 0$

$A = [-2, 4]$, $\mathbf{b} = 0 \Rightarrow$ A.M.: $M = [-2, 4, 0]$ (REF).

Apply $R3(1, -1/2) \Rightarrow$ RREF: $[1, -2, 0]$. Free variable: $y = t$, pivot variable: x.

From RREF: $x = 2y = 2t \Rightarrow$ Solution $\mathbf{x} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Ex. 2: $-2x + 4y = 7$

$A = [-2, 4]$, $\mathbf{b} = 7 \Rightarrow$ A.M.: $M = [-2, 4, 7]$ (REF).

Apply $R3(1, -1/2) \Rightarrow$ RREF: $[1, -2, -7/2]$. Free: $y = t$, pivot: x.

From RREF: $x = 2y - 7/2 = 2t - 7/2 \Rightarrow$ Solution:

$$\mathbf{x} = \begin{bmatrix} 2t - 7/2 \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -7/2 \\ 0 \end{bmatrix}$$

Ex. 3: $\begin{aligned} -2x + 4y &= 7 \\ 3x - 4y &= 0 \end{aligned}$ A.M.: $M = \begin{bmatrix} -2 & 4 & 7 \\ 3 & -4 & 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & 4 & 7 \\ 3 & -4 & 0 \end{bmatrix} \overset{R3(1,-1/2)}{\longrightarrow} \begin{bmatrix} 1 & -2 & -7/2 \\ 3 & -4 & 0 \end{bmatrix} \overset{R1(2,1,-3)}{\longrightarrow} \begin{bmatrix} 1 & -2 & -7/2 \\ 0 & 2 & 21/2 \end{bmatrix} \text{(REF)}$$

$$\overset{R3(2,1/2)}{\longrightarrow} \begin{bmatrix} 1 & -2 & -7/2 \\ 0 & 1 & 21/4 \end{bmatrix} \overset{R1(1,2,2)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 21/4 \end{bmatrix} \text{(RREF)} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 21/4 \end{bmatrix}$$

Ex. 4: $\begin{array}{rcl} -3x-6y-3z &=& -3 \\ -6x+4y+z &=& -8 \end{array}$ A.M.: $M = \begin{bmatrix} -3 & -6 & -3 & -3 \\ -6 & 4 & 1 & -8 \end{bmatrix}$

$$\begin{bmatrix} -3 & -6 & -3 & -3 \\ -6 & 4 & 1 & -8 \end{bmatrix} \overset{R3(1,-1/3)}{\longrightarrow} \begin{bmatrix} 1 & 2 & 1 & 1 \\ -6 & 4 & 1 & -8 \end{bmatrix} \overset{R1(2,1,6)}{\longrightarrow} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 16 & 7 & -2 \end{bmatrix} \text{(REF)}$$

$$\overset{R3(2,1/16)}{\longrightarrow} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 7/16 & -1/8 \end{bmatrix} \overset{R1(1,2,-2)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1/8 & 5/4 \\ 0 & 1 & 7/16 & -1/8 \end{bmatrix} \text{(RREF)}$$

Free: $x_3 = t$, pivots: $x_1 = 5/4 - t/8$, $x_2 = -1/8 - 7t/16$.

$$\mathbf{x} = \begin{bmatrix} 5/4 - t/8 \\ -1/8 - 7t/16 \\ t \end{bmatrix} = t \begin{bmatrix} -1/8 \\ -7/16 \\ 1 \end{bmatrix} + \begin{bmatrix} 5/4 \\ -1/8 \\ 0 \end{bmatrix}$$

Ex. 5: Solve $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 7 & -4 & 5 \\ 4 & -3 & -3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \text{A.M.: } M = \begin{bmatrix} 3 & -3 & 1 & 0 \\ 7 & -4 & 5 & 3 \\ 4 & -3 & -3 & 1 \end{bmatrix}$$

$$M \overset{R3(1,1/3)}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1/3 & 0 \\ 7 & -4 & 5 & 3 \\ 4 & -3 & -3 & 1 \end{bmatrix} \overset{R1(2,1,-7),\,R1(3,1,-4)}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1/3 & 0 \\ 0 & 3 & 8/3 & 3 \\ 0 & 1 & -13/3 & 1 \end{bmatrix}$$

$$\overset{R3(2,1/3)}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1/3 & 0 \\ 0 & 1 & 8/9 & 1 \\ 0 & 1 & -13/3 & 1 \end{bmatrix} \overset{R1(3,2,-1)}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1/3 & 0 \\ 0 & 1 & 8/9 & 1 \\ 0 & 0 & -47/9 & 0 \end{bmatrix}$$

Now use back-substitution:

3rd row $\Rightarrow x_3 = 0$

2nd row $\Rightarrow x_2 = 1$

1st row $\Rightarrow x_1 = 1$

$$\Rightarrow \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Matlab's rref–Command

Ex. 5: Solve $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} -7 & -4 & -5 & -9 \\ 1 & 10 & 7 & 10 \\ -2 & 0 & -3 & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 31 \\ -18 \\ -2 \end{bmatrix}; \ \text{A.M.:} \begin{bmatrix} -7 & -4 & -5 & -9 & 31 \\ 1 & 10 & 7 & 10 & -18 \\ -2 & 0 & -3 & 0 & -2 \end{bmatrix}$$

Floating point format:

```
M=[-7 -4 -5 -9 31;1 10 7 10 -18;-2 0 -3 0 -2];
rref(M)
1.0000  0         0           0.9740   -4.9221
0       1.0000    0           1.3571   -4.0714
0       0         1.0000     -0.6494    3.9481
```

Rational format:

```
format rat,rref(M)
1       0         0         75/77     -379/77
0       1         0         19/14     -57/14
0       0         1        -50/77      304/77
```

Symbolic computation:

```
M=sym(M);rref(M)
[1,0,0,75/77,-379/77]
[0,1,0,19/14, -57/14]
[0,0,1,-50/77,304/77]
```

Solution:

$$\mathbf{x} = t \begin{bmatrix} -75/77 \\ -19/14 \\ 50/77 \\ 1 \end{bmatrix} + \begin{bmatrix} -379/77 \\ -57/14 \\ 304/77 \\ 0 \end{bmatrix}$$