7.2: Linear Systems with Two or Three Variables

I. Geometry of Solutions

I.1 One equation for 2 unknowns

Ex.: $$2x + y = 2 \qquad (1)$$

Solutions form straight line:

$$y = 2 - 2x \qquad (2)$$

Vector form of (2):

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2-2x \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

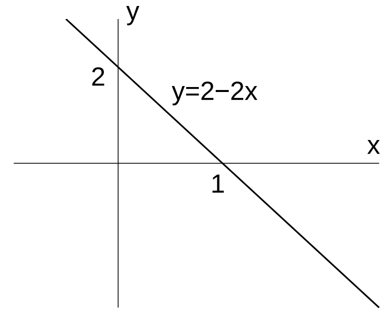
Set

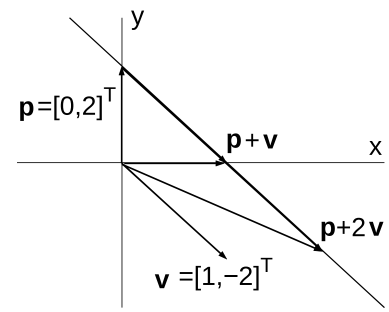
$$\mathbf{p} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Every vector of the family

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{p} + t\mathbf{v} = \begin{bmatrix} t \\ 2-2t \end{bmatrix}$$

(arbitrary t) satisfies (1), since





$$2t + (2 - 2t) = 2$$

Matrix-vector form of (1):

$$[2, 1] \begin{bmatrix} x \\ y \end{bmatrix} = 2$$

Note:

$$[2, 1]\mathbf{p} = [2, 1] \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2$$

$$[2, 1]\mathbf{v} = [2, 1] \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$$

Every point on the straight line (2) can be written as $\mathbf{p} + t\mathbf{v}$ for some t.

I.2 Two equations for 2 unknowns

Ex.:
$$\begin{array}{rcl} x+y &=& 2 \\ x-y &=& 0 \end{array} \qquad (3)$$

Solutions are intersections of two non-parallel lines ⇒ single point: (see p.4 for solution)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Possible intersections of two lines:

- Lines intersect in a single point ⇒ unique solution (generic)
- Lines are parallel, but not coincident ⇒ solutions don't exist

  Ex.
  $$\begin{array}{rcrcl} x &+& y &=& 2 \\ 2x &+& 2y &=& 3 \end{array}$$

- Lines coincide ⇒ solutions form straight line

  Ex.
  $$\begin{array}{rcrcl} x &+& y &=& 2 \\ 2x &+& 2y &=& 4 \end{array}$$

  Solutions: $y = 2 - x$

I.3 One equation for 3 unknowns

Ex.: $2x - 2y + z = 2 \qquad (4)$

Solutions form plane: $z = 2 - 2x + 2y$

Vector form of solutions:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Use parameters (s, t):

$$\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w} \qquad (5)$$

$$\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Any solution of (4) can be represented in the form (5) for some (s, t)

Matrix-vector form of (4):

$$[2, -2, 1] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2$$

$[2, -2, 1]\mathbf{v} = [2, -2, 1]\mathbf{w} = 0$

$[2, -2, 1]\mathbf{p} = 2$

I.4 Two equations for 3 unknowns

Solutions are intersections of two planes

Possibilities:

- Planes intersect in a line (generic situation)
- Planes are parallel but not coincident $\Rightarrow$ no solutions
- Planes coincide $\Rightarrow$ solutions form a plane

I.5 Three equations for 3 unknowns

Solutions are intersections of three planes

Possibilities: Planes intersect

- in a single point (generic)
- in a line (two planes coincide)
- in a plane (all three planes coincide)
- not at all (two planes are parallel)

I.6 m equations for n unknowns

If $m \leq n$,

- there may be no solutions
- the minimum number, k, of parameters needed to capture *all* solutions satisfies $k \geq n - m$
- generic situation: $k = n - m$

II. Solution of Linear Systems of Equations

Ex. II.1 Solve system (3):

$$\begin{aligned} x + y &= 2 \\ x - y &= 0 \end{aligned}$$

(1) Subtract 1st equation from 2nd equation

$$\Rightarrow \left\{ \begin{array}{rcrcr} x & + & y & = & 2 \\ & - & 2y & = & -2 \end{array} \right\}$$

(2) Multiply 2nd equation by $-1/2$

$$\Rightarrow \left\{ \begin{array}{rcrcr} x & + & y & = & 2 \\ & & y & = & 1 \end{array} \right\}$$

(3) Subtract 2nd equation from 1st equation (or sub y)

$$\Rightarrow \left\{ \begin{array}{rcr} x & = & 1 \\ y & = & 1 \end{array} \right\}$$

Matrix Procedure: Rewrite (3) as

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \left(A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right)$$

Form augmented matrix:

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = [A, \mathbf{b}]$$

(1) Subtract 1st row from 2nd row

$$\Rightarrow M \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \end{bmatrix} = M_1$$

(2) Multiply 2nd row by $-1/2$

$$\Rightarrow M_1 \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} = M_2$$

(3) Subtract 2nd row from 1st row

$$\Rightarrow M_2 \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = M_3$$

Rewrite M_3 as system of equations:

$$\left\{ \begin{array}{rcrcr} x & & & = & 1 \\ & & y & = & 1 \end{array} \right\}$$

Ex. II.2:
2 equations for 3 unknowns

$$\begin{array}{rcrcrcr} & u & - & 4v & + & w & = & -2 \\ - & 2u & + & 10v & - & 3w & = & 4 \end{array}$$

Matrix–vector form: $A\mathbf{u} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -4 & 1 \\ -2 & 10 & -3 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Augmented matrix:

$$M = \begin{bmatrix} 1 & -4 & 1 & -2 \\ -2 & 10 & -3 & 4 \end{bmatrix} = [A, \mathbf{b}]$$

(1) Add 2 × 1st row to 2nd row (to eliminate 1st entry in 2nd row)

$$M \to \begin{bmatrix} 1 & -4 & 1 & -2 \\ 0 & 2 & -1 & 0 \end{bmatrix} = M_1$$

(2) Multiply 2nd row by 1/2 to normalize 1st nonzero entry of 2nd row to unity

$$M_1 \to \begin{bmatrix} 1 & -4 & 1 & -2 \\ 0 & 1 & -1/2 & 0 \end{bmatrix} = M_2$$

(3) Add 4 × 2nd row to 1st row (eliminate v from 1st equation)

$$M_2 \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1/2 & 0 \end{bmatrix} = M_3$$

Rewrite M_3 as system of equations:

$$\left\{ \begin{array}{ccccc} u & & - & w & = & -2 \\ & v & - & w/2 & = & 0 \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{ccc} u & = & -2 + w \\ v & = & w/2 \end{array} \right\}$$

w can be chosen arbitrarily. Set $w = t$ and write solution in vector form:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 + t \\ t/2 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

Ex. II.3:
3 equations for 3 unknowns

$$\begin{array}{rcrcrcr} & x & & & + & 3z & = & -2 \\ - & 3x & + & 2y & - & 5z & = & 2 \\ & 2x & - & 4y & + & z & = & 1 \end{array}$$

Matrix–vector form: $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & -5 \\ 2 & -4 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Augmented matrix:

$$M = \begin{bmatrix} 1 & 0 & 3 & -2 \\ -3 & 2 & -5 & 2 \\ 2 & -4 & 1 & 1 \end{bmatrix}$$

(1) Add 3 × 1st row to 2nd row, subtract 2 × 1st row from 3rd row to create zeros below 1st entry of 1st column

$$M \to \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 2 & 4 & -4 \\ 0 & -4 & -5 & 5 \end{bmatrix} = M_1$$

(2) Add 2 × 2nd row to 3rd row to create zero in 3rd entry of 2nd column

$$M_1 \to \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 2 & 4 & -4 \\ 0 & 0 & 3 & -3 \end{bmatrix} = M_2$$

(3) Divide 2nd row by 2 and 3rd row by 3 to create 1's in the diagonal

$$M_2 \to \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} = M_3$$

(4) Subtract 2 × 3rd row from 2nd row and 3 × 3rd row from 1st row to create zeros above last entry of 3rd column (eliminate z from 1st and 2nd equations)

$$M_3 \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = M_4$$

Rewrite M_4 as system of equations

$\Rightarrow$ solution: $\left\{ \begin{array}{rcr} x & = & 1 \\ y & = & 0 \\ z & = & -1 \end{array} \right\}$