Chapter 7: Matrix Algebra

7.1 Vectors and Matrices

a_{ij}, x_i, y_i real numbers

$\mathbf{m} \times \mathbf{n}$**-matrix:**

$$\begin{aligned} A &= [a_{ij}]_{mn} \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

Column vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad (m \times 1)$$

Row vector:

$$\mathbf{y} = [y_1, y_2, \cdots, y_n] \quad (1 \times n)$$

j-th column of A:

$$\text{col}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

i-the row of A:

$$\text{row}_i(A) = [a_{i1}, a_{i2}, \cdots, a_{in}]$$

Matrix addition:

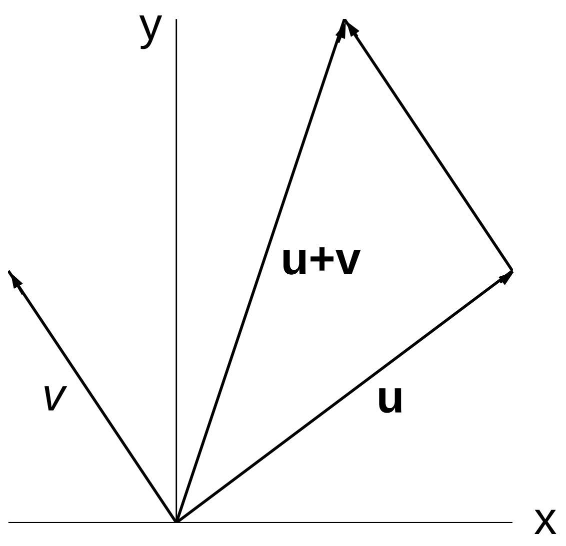
$$[a_{ij}]_{mn} + [b_{ij}]_{mn} = [a_{ij} + b_{ij}]_{mn}$$

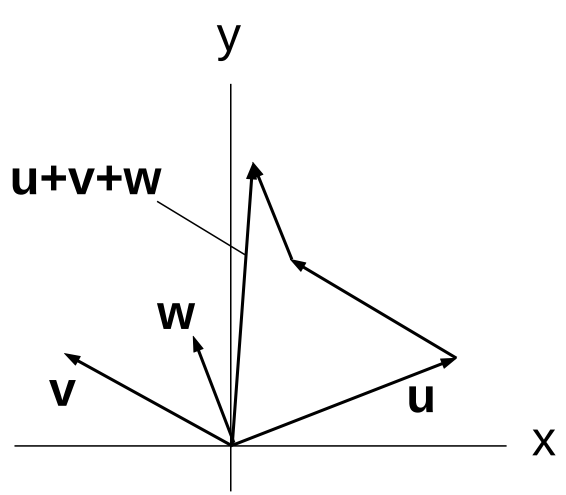
Scalar multiplication:

$$\alpha[a_{ij}]_{mn} = [\alpha a_{ij}]_{mn} \text{ } (\alpha \text{ number})$$

$\mathbf{R}^n$: set of column vectors of length n

Vector Addition





Examples:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\text{col}_2(A) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\text{row}_2(A) = [4, 5, 6]$$

$$B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

For $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow A + C$ is not defined

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

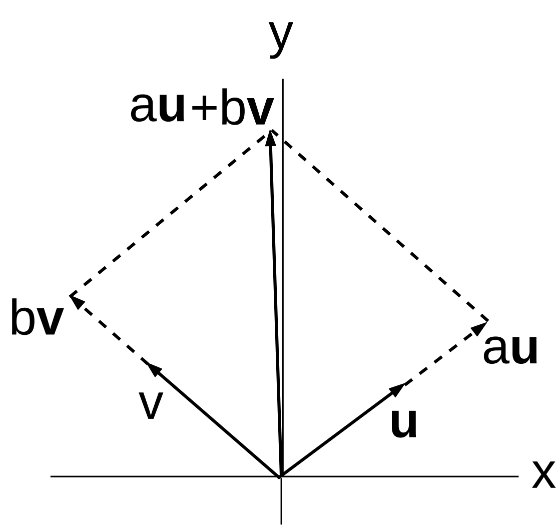
Linear combinations of vectors:

$\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$: vectors in $\mathbf{R}^n$

$a_1, a_2, \cdots, a_p$: numbers

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

is a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_p$



Row-column multiplication:

$$\mathbf{a} = [a_1, a_2, \cdots, a_n], \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\mathbf{ab} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

Transpose of $\mathbf{b}$:

$$\mathbf{b}^T = [b_1, b_2, \cdots, b_n]$$

Length of $\mathbf{b}$:

$$|\mathbf{b}| = \sqrt{\mathbf{b}^T\mathbf{b}} = \sqrt{b_1^2 + \cdots + b_n^2}$$

$$[4, 5, 6] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 = 32$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow |\mathbf{x}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

Matrix-vector multiplication:

$$A = [a_{ij}]_{mn}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

columns of A: $\rightarrow$ length m

rows of A: $\rightarrow$ length n

$$A\mathbf{x} \stackrel{\text{def}}{=} \begin{bmatrix} \text{row}_1(A)\,\mathbf{x} \\ \text{row}_2(A)\,\mathbf{x} \\ \vdots \\ \text{row}_m(A)\,\mathbf{x} \end{bmatrix}$$

Other interpretation:

$$\begin{aligned} A\mathbf{x} &= x_1\text{col}_1(A) + x_2\text{col}_2(A) \\ &\quad + \cdots + x_n\text{col}_n(A) \\ &= \text{linear combination of columns of } A \end{aligned}$$

Linearity: $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$

$A(\alpha\mathbf{x}) = \alpha A\mathbf{x}$

Example:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & -2 \end{bmatrix} \quad (2 \times 3)$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + (-3) \cdot 3 \\ 4 \cdot 1 + 0 \cdot 2 + (-2) \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -2 \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} A\mathbf{x} &= 1\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -2 \end{bmatrix} \end{aligned}$$

Linear Systems of Equations

Form:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

a_{ij}, b_i: given; x_i: sought

Matrix-vector notation:

$$A\mathbf{x} = \mathbf{b}$$

$A = [a_{ij}]_{mn}$: $m \times n$ matrix

$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$: target vector

$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$: unknown vector

Interpretation:

Find all possible linear combinations of columns of A that yield $\mathbf{b}$

Example:

$$\begin{aligned} 3x_1 + 2x_2 - 5x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 3 & 2 & -5 \\ 4 & -1 & 5 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Task: Find all $\mathbf{x}$'s s.t.

$$A\mathbf{x} = \mathbf{b}$$

or

$$x_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Matrix-matrix product

Definition

Given $A = [a_{ij}]_{mn}$, $B = [b_{ij}]_{np}$,

$$C = AB = [c_{ik}]_{mp}$$

is defined as

$$\begin{aligned} c_{ik} &= a_{i1}b_{1k} + \cdots + a_{in}b_{nk} \\ &= \text{row}_i(A)\,\text{col}_k(B) \end{aligned}$$

Other interpretation

$$\text{col}_k(AB) = A\,\text{col}_k(B)$$

or

$$AB = [A\,\text{col}_1(B), \cdots, A\,\text{col}_p(B)]$$

$$A = \begin{bmatrix} -2 & 3 & 1 \\ 9 & 8 & -2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 5 \\ 9 & 0 \\ 0 & -2 \end{bmatrix}$$

$$c_{11} = [-2, 3, 1] \begin{bmatrix} 1 \\ 9 \\ 0 \end{bmatrix} = 25$$

$$c_{12} = [-2, 3, 1] \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} = -12$$

$$c_{21} = [9, 8, -2] \begin{bmatrix} 1 \\ 9 \\ 0 \end{bmatrix} = 81$$

$$c_{22} = [9, 8, -2] \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} = 49$$

$$\begin{aligned} \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} &= A\,\text{col}_1(B) = \begin{bmatrix} 25 \\ 81 \end{bmatrix} \\ \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} &= A\,\text{col}_2(B) = \begin{bmatrix} -12 \\ 49 \end{bmatrix} \end{aligned}$$

Algebraic properties of matrix product; special matrices

Associativity

$$A(BC) = (AB)C \equiv ABC$$

Distributivity

$$\begin{aligned} A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \end{aligned}$$

However

- If AB is defined, BA is defined only if A is $m \times n$ and B is $n \times m$
- If A, B are both $n \times n$, then

$$AB \neq BA$$

in general

Special matrices

- $A = [a_{ij}]_{mn} \rightarrow A^T = [a_{ji}]_{nm}$
  A^T: transpose of A
- $n \times n$–identity matrix:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

If A is $m \times n$: $AI = A$

If B is $n \times m$: $IB = B$

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 9 & 2 \end{bmatrix}, \; B = \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ -25 & -5 \end{bmatrix}, BA = \begin{bmatrix} -9 & -5 \\ 18 & 5 \end{bmatrix}$$