6.2: Runge Kutta Methods (RKM)

(A) 2nd Order RKM (or Improved Euler Method)

Failure of Euler Method: Only slope on left end of interval $[t, t+h]$ is used.

Improvement: Given $t, y(t)$,

- compute slope at t

$$s_l = f(t, y(t))$$

- find slope at $t+h$ via EM

$$\begin{aligned} y_E &= y(t) + hs_l \\ s_r &= f(t+h, y_E) \end{aligned}$$

- approximate $y(t+h)$ via average slope

$$y(t+h) \approx y(t) + h(s_l + s_r)/2$$

Iteration Scheme

Start: y_0, t_0

For $k = 0$ to $k = N$:

$$\begin{aligned} t_{k+1} &= t_k + h \\ s_l &= f(t_k, y_k) \\ s_r &= f(t_{k+1}, y_k + hs_l) \\ y_{k+1} &= y_k + h(s_l + s_r)/2 \end{aligned}$$

Ex. Approximate the solution to

$$y' = t - y,\ y(0) = 0.5$$

in $0 \le t \le 1$ using $h = 0.25$.

Start: $y_0 = 0.5$, $t_0 = 0$

$$\begin{aligned}
t_1 &= 0.25 \\
s_l &= t_0 - y_0 = -0.5 \\
s_r &= t_1 - (y_0 + hs_l) = -0.125 \\
y_1 &= y_0 + h(s_l + s_r)/2 = 0.4219
\end{aligned}$$

$$\begin{aligned}
t_2 &= 0.5 \\
s_l &= t_1 - y_1 = -0.1719 \\
s_r &= t_2 - (y_1 + hs_l) = 0.1211 \\
y_2 &= y_1 + h(s_l + s_r)/2 = 0.4155
\end{aligned}$$

$$\begin{aligned}
t_3 &= 0.75 \\
s_l &= t_2 - y_2 = 0.0845 \\
s_r &= t_3 - (y_2 + hs_l) = 0.3134 \\
y_3 &= y_2 + h(s_l + s_r)/2 = 0.4653
\end{aligned}$$

$$\begin{aligned}
t_4 &= 1 \\
s_l &= t_3 - y_3 = 0.0845 \\
s_r &= t_4 - (y_3 + hs_l) = 0.3134 \\
y_4 &= y_3 + h(s_l + s_r)/2 = 0.4653
\end{aligned}$$

Ex.: $y' = t - y$, $y(0) = 0.5$
Approximate $y(1)$ for stepsizes

$$h = 1/m, \quad m = 1, 2, 4, 8, 16, 32$$

Exact Value: $y(1) = 0.551819$

Error: $E(h) = |y(1) - y_m|$

h	y_m	$E(h)$
1	0.75	0.198181
1/2	0.585938	0.034118
1/4	0.558794	0.006974
1/8	0.553400	0.001581
1/16	0.552196	0.000377
1/32	0.551911	0.000092

$$E(h/2) \approx E(h)/4 \ \Rightarrow\ E(h) \approx C\,h^2$$

Theorem: There $\exists\ C > 0$ s.t.

$$E(h) \le C\,h^2$$

(2nd order RKM is second order method)

(B) 4th Order RKM

Idea: Given t and $y = y(t)$, compute slopes s_1, s_2, s_3, s_4 at four carefully chosen points s.t. error is minimized.

Approximation:

$$y(t+h) \approx y + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

Iteration $\mathbf{k} \to \mathbf{k+1}$:

$$\begin{aligned} s_1 &= f(t_k, y_k) \\ s_2 &= f(t_k + h/2, y_k + hs_1/2) \\ s_3 &= f(t_k + h/2, y_k + hs_2/2) \\ s_4 &= f(t_k + h, y_k + hs_3) \\ y_{k+1} &= y_k + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4) \\ t_{k+1} &= t_k + h \end{aligned}$$

Ex.: $y' = t - y$, $y(0) = 0.5$, $y(1) \approx y_m$

$m = 1, 2, 4, 8, 16, 32$, $h = 1/m$

Exact Value: $y(1) = 0.551819162$

Error: $E(h) = |y(1) - y_m|$

h	y_m	$E(h)$
1	0.5625	0.010680838
1/2	0.552256266	0.000437105
1/4	0.551841299	0.000022137
1/8	0.551820408	0.000001246
1/16	0.551819236	0.000000074
1/32	0.551819166	0.000000005

$$E(h/2) \approx E(h)/16 \;\Rightarrow\; E(h) \approx C\,h^4$$

Theorem: There $\exists\, C > 0$ s.t.

$$E(h) \le C\,h^4$$

(4th order RKM is fourth order method)

Error Comparison

Ex. $y' = t - y,\ y(0) = 0.5$

$y(1) \approx y_m \rightarrow E(h) = |y(1) - y_m|$

$h = 1/m, \ \ m = 1, 2, 4, 8, 16, 32$

h	EM	RKM2	RKM4
1	0.5518	0.198181	0.010680838
1/2	0.1768	0.034118	0.000437105
1/4	0.0772	0.006974	0.000022137
1/8	0.0364	0.001581	0.000001246
1/16	0.0177	0.000377	0.000000074
1/32	0.0087	0.000092	0.000000005

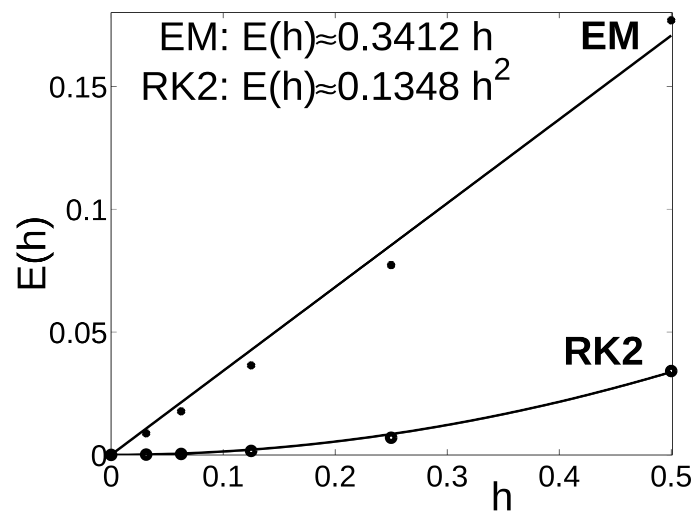
$E(h)$ for

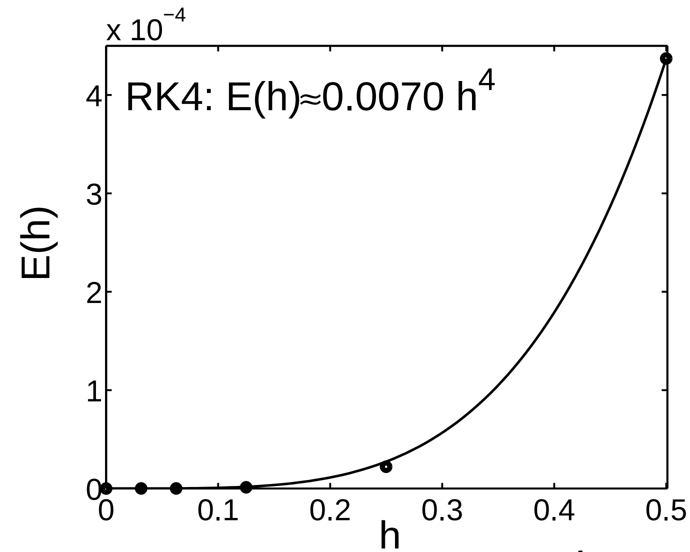
EM: Euler Method

RKM2: 2nd order RKM

RKM4: 4th order RKM

Least square fit of $\mathbf{E(h)}$





Worked out Examples from Exercises

Ex. 3: $y' = ty$, $y(0) = 1$.

Compute five RK2-iterates for $h = 0.1$.
Arrange computation and results in a table.

k	t_k	y_k	s_l	s_r	h	$h(s_l + s_r)/2$
0	0	1	0	0.1	0.1	0.005
1	0.1	1.0050	0.1005	0.2030	0.1	0.0152
2	0.2	1.0202	0.2040	0.3122	0.1	0.0258
3	0.3	1.0460	0.3138	0.4309	0.1	0.0372
4	0.4	1.0832	0.4333	0.5633	0.1	0.0498
5	0.5	1.1331	0.5665	0.7138	0.1	0.0640

Ex. 7: $z' + z = \cos x$, $z(0) = 1$

(i) Compute RK2-approximations in $0 \le x \le 1$ for $h = 0.2$, $h = 0.1$, $h = 0.05$.

(ii) Find exact solution

(iii) Plot exact solution as curve and RK2 approximations as points.

(i) In Matlab, the RK2 approximation for $h = 0.2$ is computed and stored in arrays *x0_2*, *z0_2* via

```
h=0.2;
m=1/h;x=0;z=1;
xv=x;zv=z;
for k=1:m
    sl=cos(x)-z;
    sr=cos(x+h)-(z+sl*h);
    z=z+h*(sl+sr)/2;zv=[zv z];
    x=x+h;xv=[xv x];
end
x0_2=xv;z0_2=zv;
```

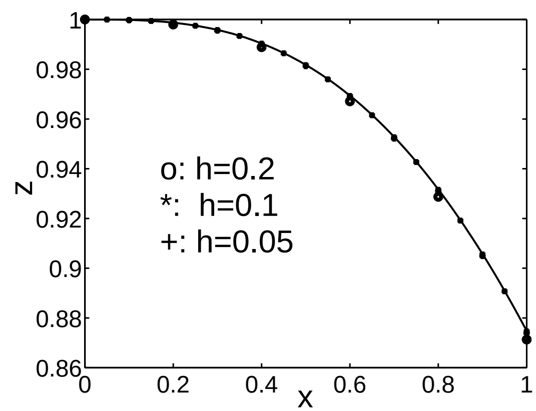
Analogously for $h = 0.1$ and $h = 0.05$ (arrays *x0_1*, *z0_1* and *x0_05*, *z0_05*).

(ii) Variation of Parameter:

$$z_h' = -z \;\Rightarrow\; z_h(x) = e^{-x}$$

$$\begin{aligned} z(x) &= e^{-x} + \int_0^x e^{\xi}\cos(\xi)\,d\xi \\ &= (\cos x + \sin x + e^{-x})/2 \end{aligned}$$

(iii) Plot:
(see CN Sec. 6.1 for commands)



Ex. 7a: $z' + z = \cos x$, $z(0) = 1$

(i) Compute RK4-approximation in $0 \leq x \leq 1$ for $h = 0.2$.

(iii) Plot exact solution as curve and RK4 approximation as points.

(i) RK4 approximation for $h = 0.2$ is computed and stored in arrays *xv*, *zv*:

```
h=0.2;
m=1/h;x=0;z=1;
xv=x;zv=z;
for k=1:m
    s1=cos(x)-z;
    s2=cos(x+h/2)-(z+s1*h/2);
    s3=cos(x+h/2)-(z+s2*h/2);
    s4=cos(x+h)-(z+s3*h);
    z=z+h*(s1+2*s2+2*s3+s4)/6;
    zv=[zv z];
    x=x+h;xv=[xv x];
end
```

(iii) Matlab plot commands:

```
x=linspace(0,1,100);
z=1/2*(cos(x)+sin(x)+exp(-x));
plot(xv,zv,'ko',x,z,'k'),
xlabel('x'),ylabel('z'),
axis([0 1 0.86 1])
```