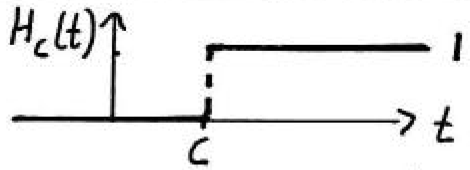
5.5 Discontinuous Forcing Functions

Heaviside step function ($c \geq 0$)

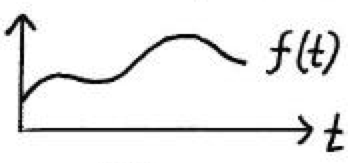


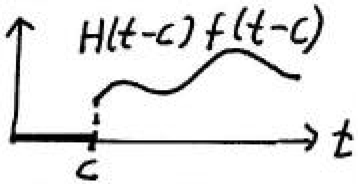
$$H_c(t) \equiv H(t-c) = \begin{cases} 1 & \text{for } t \geq c \\ 0 & \text{for } t < c \end{cases}$$

Since we restrict to $t \geq 0$, identify

$$H_0(t) \equiv H(t) = 1 \quad (\text{in } t \geq 0)$$

t-shift property of $\mathcal{L}$:





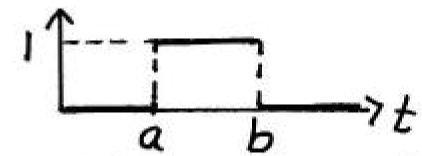
$$\mathcal{L}\{H(t-c)\,f(t-c)\} = e^{-cs}F(s)$$

or $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = H(t-c)\,f(t-c)$

$$\Rightarrow \mathcal{L}\{H_c(t)\} = e^{-cs}\mathcal{L}(1) = \frac{e^{-cs}}{s}$$

Boxcar (Interval) function

$$H_{ab}(t) = H_a(t) - H_b(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } a \leq t < b \\ 0 & \text{for } t \geq b \end{cases}$$



$(a < b)$

$$\mathcal{L}\{H_{ab}(t)\}(s) = (e^{-as} - e^{-bs})/s$$

Piecewise Defined Functions

Ex. 1: $g(t) = \begin{cases} 2t & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < \infty \end{cases}$

Use boxcars:

$$g(t) = 2t\,[1 - H(t-1)] + 2H(t-1)$$
$$= 2t - 2H(t-1)(t-1)$$

$$\Rightarrow \mathcal{L}(g)(s) = 2\mathcal{L}\{t\}(s) - 2\mathcal{L}\{H(t-1)(t-1)\}$$
$$= 2/s^2 - 2e^{-s}\mathcal{L}\{t\}(s)$$
$$= 2/s^2 - 2e^{-s}/s^2$$

Inverse $\mathcal{L}$-Transforms of Functions with Exponential Terms

Ex. 2 : $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+1)}\right\}(t)$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}(t-1)$$

$$F(s) \equiv \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-s}F(s)\}(t)$$

$$= \mathcal{L}^{-1}\{1/s^2\}(t-1) + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-1)$$

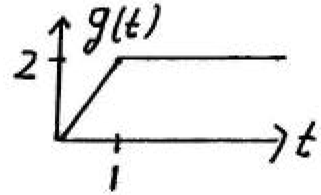
$$= H(t-1)[t-1-\sin(t-1)]$$

IVP's with Discontinuous Forcings

Ex. 3 : $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$

$$g(t) = \begin{cases} 2t & \text{for } 0 \le t < 1 \\ 2 & \text{for } t \ge 1 \end{cases}$$

(as in Ex. 1)



$$\mathcal{L}(y''+y) = (s^2+1)Y - 1$$

$$\mathcal{L}(g) = 2(1-e^{-s})/s^2 \quad \text{(from Ex. 1)}$$

$$\Rightarrow (s^2+1)Y - 1 = 2(1-e^{-s})/s^2$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{2}{s^2(s^2+1)} - \frac{2e^{-s}}{s^2(s^2+1)}$$

$$\Rightarrow y(t) = \sin t + 2(t - \sin t)$$
$$- 2H(t-1)[t-1-\sin(t-1)] \quad \text{(Ex. 2)}$$

$$\Rightarrow y(t) = -\sin t + 2t[1 - H(t-1)] + 2H(t-1)[1 + \sin(t-1)]$$

$$= (2t - \sin t)[1 - H(t-1)] + [2 + 2\sin(t-1) - \sin t]H(t-1)$$

$$= \begin{cases} 2t - \sin t & \text{for } 0 \le t < 1 \\ 2 + 2\sin(t-1) - \sin t & \text{for } t \ge 1 \end{cases}$$

(see text, p. 221, for graph)

5.6 The Delta Function

$F(t)$ force $\Rightarrow$

$I = \int_a^b F(t)\,dt$ = momentum change in $a \le t \le b$

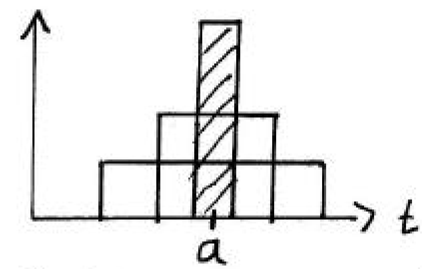
= <u>impulse</u>

$mv' = F(t) \rightarrow I = mv(b) - mv(a)$

<u>unit impulse</u> in $a - \varepsilon/2 \le t \le a + \varepsilon/2$:

$$\delta_a^\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{for } a - \varepsilon/2 \le t < a + \varepsilon/2 \\ 0 & \text{else} \end{cases}$$

$$= \frac{1}{\varepsilon}\left[H_{a-\varepsilon/2}(t) - H_{a+\varepsilon/2}(t)\right]$$



area:

$$\int_0^\infty \delta_a^\varepsilon(t)\,dt = 1$$

<u>Def.:</u> $\delta_a(t) \equiv \delta(t-a) = \lim_{\varepsilon \to 0} \delta_a^\varepsilon(t)$

Think of $\delta_a(t)$ "=" $\begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \ne a \end{cases}$

s.t. $\int_{a-\eta}^{a+\eta} \delta_a(t)\,dt = 1$ for any $\eta > 0$

$\delta_a(t)$ describes effect of "unit hammer"

"filter property":

$$\int_0^\infty \delta_a(t)\phi(t)\,dt = \phi(a)$$

for any continuous fct. $\phi(t)$

$$\Rightarrow \mathcal{L}(\delta_a)(s) = \int_0^\infty e^{-st}\delta_a(t) = e^{-sa}$$

<u>Impulse Response</u>:

$$\left.\begin{aligned} e'' + ae' + be &= \delta(t) \\ e(0) = 0,\ e'(0) &= 0 \end{aligned}\right\} (*)$$

(more precisely: $e(0-) = 0, e'(0-) = 0$)

Let $E(s) = \mathcal{L}(e)(t)$

$\Rightarrow (s^2 + as + b)E = 1$

<u>Thm.:</u> For $t > 0$ the solution of $(*)$ coincides with the solution of

$y'' + ay' + by = 0,\ y(0) = 0,\ y'(0) = 1$

5.7 Convolutions

Consider $F(s), G(s)$. Question: $\mathcal{L}^{-1}\{F(s)G(s)\} = ?$

Ex.: $F(s) = G(s) = \frac{1}{s} \Rightarrow \mathcal{L}^{-1}(FG) = \mathcal{L}^{-1}\{1/s^2\} = t$

But $\mathcal{L}^{-1}\{1/s\} = 1 \Rightarrow \mathcal{L}^{-1}\{1/s\}\mathcal{L}^{-1}\{1/s\} = 1 \neq t$

Thm.: $\mathcal{L}^{-1}\{F(s)G(s)\}(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ where $f = \mathcal{L}^{-1}(F), g = \mathcal{L}^{-1}(G)$

Def.: Given $f(t), g(t)$, the convolution of f and g is:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Properties: $f * g = g * f$, $(f * \delta_a)(t) = f(t-a)H(t-a)$

System Response: $y'' + ay' + by = f(t)$, $y(0) = 0 = y'(0)$

Apply $\mathcal{L} \Rightarrow (s^2 + as + b)Y = F(s)$

$$\Rightarrow Y(s) = \frac{1}{s^2 + as + b}F(s) = E(s)F(s)$$

where $E(s) = 1/(s^2 + as + b)$

$\Rightarrow y(t) = (e * f)(t)$ where $e(t) = \mathcal{L}^{-1}(E)(t) =$ "unit response"

$=$ response to $f(t) = \delta(t)$

(Sec. 5.6)