Ch. 5: The Laplace Transform

- Technique for solving linear DEs with constant coefficients
- Useful for discontinuous forcings

5.1 Definition and Existence of Laplace Transforms

Def.: Given a real or complex function $f(t)$, the Laplace ($\mathcal{L}$) transform of f is the following function of s:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \equiv \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,dt$$

Notation:

$$F(s) = \mathcal{L}(f)(s) = \mathcal{L}\{f(t)\}(s)$$

Def.: $f(t)$ is of exponential order if there are constants C, a s.t.

$$|f(t)| \le Ce^{at} \text{ for all } t$$

Meaning: $f(t)$ grows at most exponentially if $t \to \infty$

Ex.: e^{t^2} is *not* of exponential order

Ex.: $e^{10{,}000t}$ is of exponential order

Def.: $f(t)$ is piecewise continuous if

- in any finite interval $0 < t < T$ there are at most finitely many discontinuities
- at any point of discontinuity t_d the left and right limits $f_\mp$ exist:

$$f_-(t_d) = \lim_{t\to t_d-} f(t),\ f_+(t_d) = \lim_{t\to t_d+} f(t)$$

Ex.: $f(t) = \left\{\begin{array}{lr} 0 & \text{if } \ 0 \le t < 1 \\ e^{t-1} & \text{if } \ t \ge 1 \end{array}\right\}$ has a discontinuity at $t_d = 1$:

$$f_-(1) = 0,\ \ f_+(1) = 1$$

Thm.: If $f(t)$ is piecewise continuous in $0 \le t < \infty$ and of exponential order, then $\mathcal{L}(f)(s)$ exists for $s > a$.

Basic Example: $\mathcal{L}(1)(s) = \int_0^\infty e^{-st} 1\,dt = \lim_{T\to\infty} \int_0^T e^{-st} dt$

$$= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-st}\Big|_0^T\right] = \lim_{T\to\infty}\left[-\frac{1}{s}e^{-sT} + \frac{1}{s}\right] = \frac{1}{s} \ \text{ for } \ s > 0$$

5.2: Basic Properties of the Laplace Transform

$F(s) = \mathcal{L}\{f(t)\}(s)$, $Y(s) = \mathcal{L}\{y(t)\}(s)$

1. Linearity:

$$\mathcal{L}(af + bg)(s) = a\mathcal{L}(f)(s) + b\mathcal{L}(g)(s)$$

2. 'Reality':

$f(t)$ real $\Rightarrow$ $\mathcal{L}(f)(s)$ real

Consequence: $f(t)$ complex $\Rightarrow$

$$\begin{aligned} \text{Re}(\mathcal{L}(f)(s)) &= \mathcal{L}(\text{Re}(f))(s) \\ \text{Im}(\mathcal{L}(f)(s)) &= \mathcal{L}(\text{Im}(f))(s) \end{aligned}$$

3. Derivatives:

$$\begin{aligned} \mathcal{L}(y')(s) &= sY(s) - y(0) \\ \mathcal{L}(y'')(s) &= s^2Y(s) - sy(0) - y'(0) \\ \mathcal{L}(y^{(k)})(s) &= s^kY(s) - s^{k-1}y(0) - \\ & \quad s^{k-2}y'(0) - \cdots - y^{(k-1)}(0) \end{aligned}$$

4. Multiplication by e^{ct} $(c \in \mathbf{C})$:

$$\mathcal{L}\{e^{ct}f(t)\}(s) = F(s-c)$$

5. Multiplication by t^k:
$(k = 0, 1, 2, \ldots)$

$$\mathcal{L}\{t^k f(t)\}(s) = (-1)^k F^{(k)}(s)$$

Proof 3. for $k = 1$: Use partial integration: $\int uv'\,dt = uv - \int u'v\,dt$

$$\begin{aligned} \int_0^T e^{-st}y'(t)\,dt &= e^{-st}y(t)\Big|_0^T + s\int_0^T e^{-st}y(t)\,dt \\ &= e^{-sT}y(T) - y(0) \\ & \quad + s\int_0^T e^{-st}y(t)\,dt \end{aligned}$$

For $T \to \infty$:

$$e^{-sT}y(T) \to 0, \quad \int_0^T e^{-st}y(t)\,dt \to Y(s)$$

$$\Rightarrow \mathcal{L}(y')(s) = sY(s) - y(0)$$

Proof 4.:

$$\begin{aligned} \mathcal{L}\{e^{ct}f(t)\}(s) &= \int_0^\infty e^{-st}e^{ct}f(t)\,dt \\ &= \int_0^\infty e^{-(s-c)t}f(t)\,dt \\ &= F(s-c) \end{aligned}$$

Proof 5. for $k = 1$:
$F(s) = \int_0^\infty e^{-st}f(t)\,dt \Rightarrow$

$$F'(s) = \int_0^\infty (-t)f(t)\,dt = -\mathcal{L}\{tf(t)\}(s)$$

$\mathcal{L}$-Transforms of Functions Encountered in ODEs

ODEs with constant coefficients
$\rightarrow$ functions $t^k e^{ct}$, $k = 0, 1, 2, \ldots$

Property 5 $\Rightarrow$

$$\mathcal{L}\{t^k e^{ct}\}(s) = (-1)^k \frac{d^k}{ds^k} \mathcal{L}\{e^{ct}\}(s)$$

Property 4 $\Rightarrow$

$$\begin{aligned} \mathcal{L}\{e^{ct}\}(s) &= \mathcal{L}\{e^{ct}1\}(s) = \mathcal{L}\{1\}(s-c) \\ &= \frac{1}{s-c} \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L}\{t^k e^{ct}\}(s) &= (-1)^k \frac{d^k}{ds^k} \frac{1}{s-c} \\ &= \frac{k!}{(s-c)^{k+1}} \qquad (1) \end{aligned}$$

(1) $\Rightarrow$ **Special Transforms:**

- $k = 0$, $c \in \mathbf{R} \Rightarrow \mathcal{L}\{e^{ct}\}(s) = \frac{1}{s-c}$
- $k = 0$, $c = i\omega \Rightarrow$

  $\mathcal{L}\{e^{i\omega t}\}(s) = \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2} \Rightarrow$

  $\mathcal{L}\{\cos \omega t\}(s) = \mathrm{Re}\left(\frac{s+i\omega}{s^2+\omega^2}\right) = \frac{s}{s^2+\omega^2}$

  $\mathcal{L}\{\sin \omega t\}(s) = \mathrm{Im}\left(\frac{s+i\omega}{s^2+\omega^2}\right) = \frac{\omega}{s^2+\omega^2}$
- $k = 0$, $c = \alpha + i\beta \Rightarrow$

  $\mathcal{L}\{e^{\alpha t} e^{i\beta t}\}(s) = \frac{1}{s-\alpha-i\beta} = \frac{s-\alpha+i\beta}{(s-\alpha)^2+\beta^2}$

  $\Rightarrow \mathcal{L}\{e^{\alpha t} \cos \beta t\}(s) = \frac{s-\alpha}{(s-\alpha)^2+\beta^2}$

  $\mathcal{L}\{e^{\alpha t} \sin \beta t\}(s) = \frac{\beta}{(s-\alpha)^2+\beta^2}$

Table of $\mathcal{L}$–Transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$
1	$\frac{1}{s}$
t^k	$\frac{k!}{s^{k+1}}$
e^{ct}	$\frac{1}{s-c}$
$t^k e^{ct}$	$\frac{k!}{(s-c)^{k+1}}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$e^{\alpha t} \cos \beta t$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$
$e^{\alpha t} \sin \beta t$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$

5.3 Inverse Laplace Transform

Thm.: If $f(t)$ and $g(t)$ are piecewise continuous on $0 \le t < \infty$ and $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$ for $s > a$, then $f(t) = g(t)$ for all t in $0 \le t < \infty$ at which $f(t)$ is continuous.

$\mathcal{L}$–transform pairs:

- $f(t)$ determines $F(s)$ uniquely in $s > a$
- $F(s)$ determines $f(t)$ uniquely in $0 \le t < \infty$ except at discontinuity points.

Def.: Given $F(s)$ and $f(t)$ s.t. $F(s) = \mathcal{L}(f)(s)$, then $f(t)$ is called the inverse Laplace ($\mathcal{L}$) transform of $F(s)$, and is denoted by

$$f(t) = \mathcal{L}^{-1}(F)(t) = \mathcal{L}^{-1}\{F(s)\}(t)$$

$F(s)$	$\mathcal{L}^{-1}\{F(s)\}(t)$
$\frac{1}{s-c}$	e^{ct}
$\frac{1}{(s-c)^k}$	$\frac{t^{k-1}}{(k-1)!}e^{ct}$
$\frac{1}{(s-\alpha)^2+\beta^2}$	$\frac{e^{\alpha t}\sin\beta t}{\beta}$
$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$	$e^{\alpha t}\cos\beta t$

Inverse $\mathcal{L}$–Transform of Rational Functions

Form: $F(s) = \frac{P(s)}{Q(s)}$

- $P(s)$, $Q(s)$: polynomials
- degree of P < degree of Q

Assume $Q(s)$ has k **distinct roots**

Partial Fraction Decomposition (PFD):

$$F(s) = \sum_{\{\lambda\}} F_\lambda(s)$$

$F_\lambda(s)$: contribution from root λ

Linearity $\Rightarrow$

$$\mathcal{L}^{-1}(F)(t) = \sum_{\{\lambda\}} \mathcal{L}^{-1}(F_\lambda)(t)$$

Forms, Inverse Transforms, and Computation of $\mathrm{F}_\lambda(\mathrm{s})$

Let m be the multiplicity of λ. Set $Q_\lambda(s) = Q(s)/(s-\lambda)^m \Rightarrow Q_\lambda(\lambda) \neq 0$

Simple Root: $(m = 1)$

$$F_\lambda(s) = \frac{A}{s-\lambda}, \quad A = \frac{P(\lambda)}{Q_\lambda(\lambda)}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda)(t) = Ae^{\lambda t}$$

Complex Case: Assume $\lambda = \alpha + i\beta$, $\overline{\lambda} = \alpha - i\beta$ are a complex conjugate pair of simple roots

$$\Rightarrow F_\lambda(s) + F_{\overline{\lambda}}(s) = \frac{A}{s-\lambda} + \frac{\overline{A}}{s-\overline{\lambda}}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda + F_{\overline{\lambda}})(t) = Ae^{\lambda t} + \overline{A}e^{\overline{\lambda} t} = 2\mathrm{Re}(Ae^{\lambda t})$$

Real version: let $A = a + ib$

$$\Rightarrow F_\lambda(s) + F_{\overline{\lambda}}(s) = \frac{2a(s-\alpha) - 2b\beta}{(s-\alpha)^2 + \beta^2}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda + F_{\overline{\lambda}})(t) = 2e^{\alpha t}(a\cos\beta t - b\sin\beta t)$$

Multiple Root: $(m > 1)$

$$F_\lambda(s) = \frac{A_m}{s-\lambda} + \frac{A_{m-1}}{(s-\lambda)^2} + \cdots + \frac{A_2}{(s-\lambda)^{m-1}} + \frac{A_1}{(s-\lambda)^m}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda)(s) = e^{\lambda t}[A_m + A_{m-1}t + \cdots + A_1 t^{m-1}/(m-1)!]$$

Coefficients:

$$A_j = \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{ds^{j-1}}\left(\frac{P(s)}{Q_\lambda(s)}\right)\right]_{s=\lambda}$$

For multiple complex pairs $\lambda, \overline{\lambda}$:

$$\mathcal{L}^{-1}(F_\lambda + F_{\overline{\lambda}})(t) = 2\Big[\mathrm{Re}(A_m e^{\lambda t}) + t\,\mathrm{Re}(A_{m-1}e^{\lambda t}) + \cdots + \frac{t^{m-2}\mathrm{Re}(A_2 t e^{\lambda t})}{(m-2)!} + \frac{t^{m-1}\mathrm{Re}(A_1 t e^{\lambda t})}{(m-1)!}\Big]$$

For $\mathbf{m = 2}$:

$$A_1 = \frac{P(\lambda)}{Q_\lambda(\lambda)}, \quad A_2 = \left[\frac{d}{ds}\left(\frac{P(s)}{Q_\lambda(s)}\right)\right]_{s=\lambda}$$

Ex. 1: $F(s) = \frac{s+9}{s^2-2s-3} = \frac{s+9}{(s+1)(s-3)}$

Roots: $\lambda_1 = -1$, $\lambda_2 = 3 \rightarrow$

$$F(s) = F_{-1}(s) + F_3(s)$$

$$F_{-1}(s) = \frac{A}{s+1}, \quad F_3(s) = \frac{B}{s-3}$$

$$Q_{-1}(s) = \frac{(s+1)(s-3)}{s+1} = s-3$$

$$\Rightarrow \; A = \frac{s+9}{s-3}\Big|_{s=-1} = -2$$

$$Q_3(s) = \frac{(s+1)(s-3)}{s-3} = s+1$$

$$\Rightarrow \; B = \frac{s+9}{s+1}\Big|_{s=3} = 3$$

$$\Rightarrow \; F(s) = \frac{-2}{s+1} + \frac{3}{s-3}$$

$$\Rightarrow \; \mathcal{L}^{-1}(t) = -2e^{-t} + 3e^{3t}$$

Other methods for finding A, B:
(see text, Sec. 5.3, Example 3.6)

$$\frac{s+9}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}$$

$$\Rightarrow \; s+9 = A(s-3) + B(s+1) \quad (2)$$

Substitution method:
Substitute two values for s in (2):

$$s = 3 \quad \Rightarrow \quad 12 = 4B \; \Rightarrow \; B = 3$$
$$s = -1 \quad \Rightarrow \quad 8 = -4A \; \Rightarrow \; A = -2$$

Coefficient method: Rewrite (2) as

$$s+9 = (A+B)s + (-3A+B)$$

Equate coefficients of powers of s:

$$\Rightarrow \left\{ \begin{array}{ccc} 1 & = & A+B \\ 9 & = & -3A+B \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{ccc} A & = & -2 \\ B & = & 3 \end{array} \right\}$$

Ex. 2:

$$Y(s) = \frac{s-2}{s^2-2s-3} = \frac{s-2}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}$$

$$A = \frac{s-2}{s-3}\Big|_{s=-1} = \frac{3}{4}$$

$$B = \frac{s-2}{s+1}\Big|_{s=3} = \frac{1}{4}$$

$$\Rightarrow\ Y(s) = \frac{1}{4}\Big(\frac{3}{s+1} + \frac{1}{s-3}\Big)$$

$$\Rightarrow\ \mathcal{L}^{-1}(Y)(t) = \frac{1}{4}(3e^{-t} + e^{3t})$$

Ex. 3: $F(s) = \frac{1}{s^2+4s+13} = \frac{1}{(s+2)^2+9}$

This is of the form

$$\frac{1}{(s-\alpha)^2+\beta^2}\ \ (\alpha = -2,\ \beta = 3)$$

with inverse transform (see table)

$$(1/\beta)e^{\alpha t}\sin\beta t$$

$$\Rightarrow\ \mathcal{L}^{-1}(F)(t) = (1/3)e^{-2t}\sin 3t$$

See text, Sec. 5.3, Example 3.6, for coefficient and substitution methods.

Ex. 4: $F(s) = \frac{2s^2+s+13}{(s-1)[(s+1)^2+4]}$

(see text, Sec. 5.3, Example 3.9)

$(s+1)^2+4 = (s+1+2i)(s+1-2i)$

$\Rightarrow$ roots of $Q(s)$:

$$\lambda_1 = 1,\ \lambda_2 = -1+2i,\ \lambda_3 = \overline{\lambda_2}$$

$$F_{\lambda_1}(s) = \frac{A}{s-1},\ A = \frac{2s^2+s+13}{(s+1)^2+4}\Big|_{s=1} = 2$$

$$\Rightarrow\ \mathcal{L}^{-1}(F_{\lambda_1})(t) = 2e^t$$

Work on λ_2: $F_{\lambda_2}(s) = \frac{B}{s+1-2i}$

$$B = \frac{2s^2+s+13}{(s-1)(s+1+2i)}\Big|_{s=-1+2i} = \frac{2(1-4i-4)+(-1+2i)+13}{(-2+2i)4i} = \frac{6-6i}{-8-8i} = -\frac{3}{4}\frac{1-i}{1+i} = \frac{3i}{4}$$

$$\Rightarrow F_{\lambda_2}(s) + F_{\overline{\lambda_2}}(s) = \tfrac{3}{4}\Big(\tfrac{i}{s+1-2i} - \tfrac{i}{s+1+2i}\Big) = \tfrac{-3}{(s+1)^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}(F_{\lambda_2} + F_{\overline{\lambda_2}})(t) = -\tfrac{3}{2}e^{-t}\sin 2t$$

$$\Rightarrow\ \mathcal{L}^{-1}(F)(t) = 2e^t - (3/2)e^{-t}\sin 2t$$

Ex. 5: $Y(s) = \frac{s^2+s+4}{(s^2+1)(s^2+4)}$

$$\left.\begin{array}{rcl} s^2+1 &=& (s-i)(s+i) \\ s^2+4 &=& (s-2i)(s+2i) \end{array}\right\} \Rightarrow \text{roots:}$$

$$\lambda_1 = i,\ \lambda_2 = -i\ \lambda_3 = 2i,\ \lambda_4 = -2i$$

$$Y_{\lambda_1}(s) = \frac{A}{s-i}$$

$$\begin{aligned} A &= \left.\frac{s^2+s+4}{(s+i)(s^2+4)}\right|_{s=i} \\ &= \frac{3+i}{6i} = \frac{1}{6}(1-3i) \end{aligned}$$

$$\Rightarrow \mathcal{L}^{-1}(Y_{\lambda_1} + Y_{\overline{\lambda}_1})(t) = 2\text{Re}(Ae^{it})$$

$$= \frac{1}{3}(\cos t + 3\sin t)$$

$$Y_{\lambda_3}(s) = \frac{B}{s-2i}$$

$$\begin{aligned} A &= \left.\frac{s^2+s+4}{(s^2+1)(s+2i)}\right|_{s=2i} \\ &= \frac{2i}{(-3)4i} = -\frac{1}{6} \end{aligned}$$

$$\Rightarrow \mathcal{L}^{-1}(Y_{\lambda_3} + Y_{\overline{\lambda}_3})(t) = 2\text{Re}(Be^{2it})$$

$$= -\frac{1}{3}\cos 2t$$

$$\mathcal{L}^{-1}(Y)(t) = (1/3)(\cos t + 3\sin t - \cos 2t)$$

Ex. 6: $Y(s) = \frac{1}{(s+1)(s-1)^2}$

Roots: $\lambda_1 = -1,\ \lambda_2 = 1\ (m = 2)$

$$Y_{-1}(s) = \frac{A}{s+1},\ A = \left.\frac{1}{(s-1)^2}\right|_{s=-1} = \frac{1}{4}$$

$$Y_1(s) = \frac{B_1}{(s-1)^2} + \frac{B_2}{s-1}$$

$$\begin{aligned} B_1 &= \left.\frac{1}{s+1}\right|_{s=1} = \frac{1}{2} \\ B_2 &= \left.\left(\frac{d}{ds}\frac{1}{s+1}\right)\right|_{s=1} = -\frac{1}{4} \end{aligned}$$

$$Y(s) = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}$$

$$\mathcal{L}^{-1}(Y)(t) = \frac{1}{4}(e^{-t} - e^t + 2te^t)$$

Ex. 7: $Y(s) = \frac{s}{(s^2+2s+2)(s^2+4)}$

$Q(s) = [(s+1)^2+1](s^2+4)$: factorize $(s+1)^2+1 = (s+1-i)(s+1+i)$, $s^2+4 = (s-2i)(s+2i) \Rightarrow$ roots $\lambda_1 = -1+i$, $\lambda_2 = \overline{\lambda}_1$, $\lambda_3 = 2i$, $\lambda_4 = \overline{\lambda}_3$

$$Y_{\lambda_1}(s) = \frac{A}{s+1-i}, \quad A = \frac{s}{(s+1+i)(s^2+4)}\Big|_{s=-1+i} = \frac{-1+i}{2i((1-i)^2+4)}$$

$$= \frac{-1+i}{2i(4-2i)} = \frac{1}{4}\frac{-1+i}{1+2i} = \frac{1}{4}\frac{1}{5}(-1+i)(1-2i) = \frac{1}{20}(1+3i)$$

$$\Rightarrow \mathcal{L}^{-1}(Y_{\lambda_1} + Y_{\overline{\lambda}_1})(t) = 2e^{-t}\mathrm{Re}(\frac{1}{20}(1+3i)e^{it}) = \frac{1}{10}e^{-t}(\cos t - 3\sin t)$$

$$Y_{\lambda_3}(s) = \frac{B}{s-2i}, \quad B = \frac{s}{(s^2+2s+2)(s+2i)}\Big|_{s=2i} = \frac{2i}{(-2+4i)4i}$$

$$= -\frac{1}{4}\frac{1}{1-2i} = -\frac{1}{20}(1+2i)$$

$$\Rightarrow \mathcal{L}^{-1}(Y_{\lambda_3} + Y_{\overline{\lambda}_3})(t) = 2\mathrm{Re}(-\frac{1}{20}(1+2i)e^{2it}) = -\frac{1}{10}(\cos 2t - 2\sin 2t)$$

$$\Rightarrow \mathcal{L}^{-1}(Y)(t) = \frac{1}{10}(e^{-t}\cos t - 3e^{-t}\sin t - \cos 2t + 2\sin 2t)$$

5.4 Using the $\mathcal{L}-$Transform to Solve ODEs

Basic Idea: $\left\{\begin{array}{c}\text{IVP}\\ \text{for } y(t):\\ \text{ODE+IC}\end{array}\right\} \xrightarrow{\mathcal{L}} \left\{\begin{array}{c}\text{algebraic}\\ \text{equation}\\ \text{for } Y(s)\end{array}\right\} \xrightarrow{\text{solve}} Y(s) \xrightarrow{\mathcal{L}^{-1}} y(t)$

Ex. 8: $y'' + y = \cos 2t$

$y(0) = 0,\ y'(0) = 1$

$\mathcal{L}-$transform ODE:

$$\mathcal{L}(y'' + y) = \mathcal{L}\{\cos 2t\}$$

$$\begin{aligned}\mathcal{L}(y'') &= s^2Y - sy(0) - y'(0)\\ &= s^2Y - 1\\ \mathcal{L}(y) &= Y\end{aligned}$$

$$\Rightarrow\ \mathcal{L}(y'' + y) = (s^2 + 1)Y - 1$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2 + 4}$$

$$\Rightarrow\ (s^2 + 1)Y - 1 = \frac{s}{s^2 + 4}$$

$$\Rightarrow\ Y(s) = \frac{1}{s^2 + 1} + \frac{s}{(s^2 + 1)(s^2 + 4)}$$

$$= \frac{s^2 + s + 4}{(s^2 + 1)(s^2 + 4)}$$

$y(t) = \mathcal{L}^{-1}(Y)(t)$. From Ex. 5 $\Rightarrow$

$$y(t) = \frac{1}{3}(\cos t + 3\sin t - \cos 2t)$$

Ex. 9: $y'' - 2y' - 3y = 0$

$y(0) = 1,\ y'(0) = 0$

$$\begin{aligned}\mathcal{L}(y'') &= s^2Y - s\\ \mathcal{L}(y') &= sY - 1\end{aligned}$$

$$\Rightarrow \mathcal{L}(y'' - 2y' - 3y) = (s^2 - 2s - 3)Y - s + 2 = 0$$

$$\Rightarrow\ Y(s) = \frac{s - 2}{s^2 - 2s - 3},\ y(t) = \mathcal{L}^{-1}(Y)(t)$$

From Ex. 2: $y(t) = (1/4)(e^{3t} + 3e^{-t})$

Ex. 10: $y'' - y = e^t,\ y(0) = y'(0) = 0$

$$\mathcal{L}(y'' - y) = (s^2 - 1)Y,\ \mathcal{L}\{e^t\} = \frac{1}{s - 1}$$

$\Rightarrow (s^2 - 1)Y = 1/(s - 1)$

$\Rightarrow Y(s) = \frac{1}{(s^2-1)(s-1)} = \frac{1}{(s+1)(s-1)^2}$

Ex. 6 $\Rightarrow y(t) = (e^{-t} - e^t + 2te^t)/4$

Ex. 11: $y'' + 2y' + 2y = \cos 2t$, $y(0) = 0$, $y'(0) = 1$

$$\mathcal{L}(y'' + 2y' + 2y) = (s^2Y - 1) + 2(sY) + 2Y = (s^2 + 2s + 2)Y - 1$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4} \Rightarrow (s^2 + 2s + 2)Y - 1 = \frac{s}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+2s+2} + \frac{s}{(s^2+2s+2)(s^2+4)}$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2+2s+2}\} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2+1}\} = e^{-t}\sin t$$

From Ex. 7: $\mathcal{L}^{-1}\{\frac{s}{(s^2+2s+2)(s^2+4)}\}$

$$= \frac{1}{10}(e^{-t}\cos t - 3e^{-t}\sin t - \cos 2t + 2\sin 2t)$$

$$\Rightarrow y(t) = \frac{1}{10}(e^{-t}\cos t + 7e^{-t}\sin t - \cos 2t + 2\sin 2t)$$