4.7: Forced Harmonic Motion

Periodically forced mass-spring system: $mx'' + \mu x' + kx = F_0 \cos \omega t$

or $\quad x'' + dx' + \omega_0^2 x = A \cos \omega t$

where $d = \mu/m$, $\omega_0 = \sqrt{k/m}$, $A = F_0/m$

(A) Undamped Case

$$x'' + \omega_0^2 x = A \cos \omega t \qquad (1)$$

Try particular solution:

$$x_p(t) = a \cos \omega t + b \sin \omega t \quad \Rightarrow$$

$$x_p'' + \omega_0^2 x_p = (\omega_0^2 - \omega^2)(a \cos \omega t + b \sin \omega t)$$

The r.h.s. is equal to $A \cos \omega t$ if

$$(\omega_0^2 - \omega^2) a = A, \quad (\omega_0^2 - \omega^2) b = 0$$

$\Rightarrow a = A/(\omega_0^2 - \omega^2)$, $b = 0 \Rightarrow$

$$x_p(t) = [A/(\omega_0^2 - \omega^2)] \cos \omega t \qquad (2)$$

To find general solution, add general solution of

$$x'' + \omega_0^2 x = 0 \qquad (3)$$

F.S.S. for (3): $\cos \omega_0 t, \sin \omega_0 t$

$\Rightarrow$ general solution of (1):

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + x_p(t)$$

Beats. Assume IC: $x(0) = 0, x'(0) = 0$

$$\Rightarrow c_1 = -A/(\omega_0^2 - \omega^2), \ c_2 = 0 \ \Rightarrow$$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2} (\cos \omega t - \cos \omega_0 t) \qquad (4)$$

Set $\delta = (\omega_0 - \omega)/2$, $\overline{\omega} = (\omega_0 + \omega)/2$

Use ($\alpha = \omega t$, $\beta = \omega_0 t$)

$$\cos \alpha - \cos \beta = 2 \sin(\frac{\beta - \alpha}{2}) \sin(\frac{\beta + \alpha}{2})$$

$$\Rightarrow x(t) = \frac{A \sin \delta t}{2 \overline{\omega} \delta} \sin \overline{\omega} t \qquad (5)$$

If $\delta << \overline{\omega} \Rightarrow [A/(2\overline{\omega}\delta)] \sin \delta t$ is slowly varying envelope

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Ex.: $A = 23$, $\omega_0 = 11$, $\omega = 12$

$$\begin{aligned} \Rightarrow x(t) &= \cos 11t - \cos 12t \\ &= 2 \sin(t/2) \sin(23t/2) \end{aligned}$$

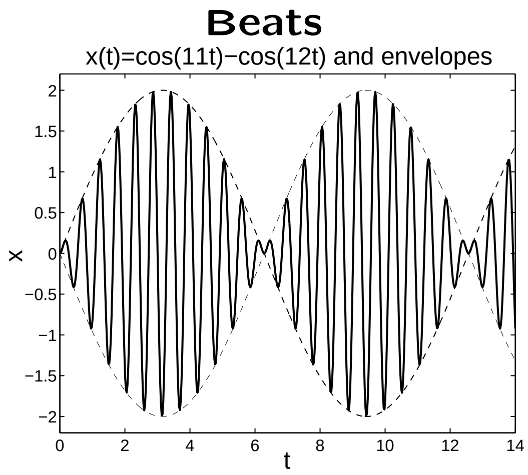
Resonant Case: $\omega = \omega_0$

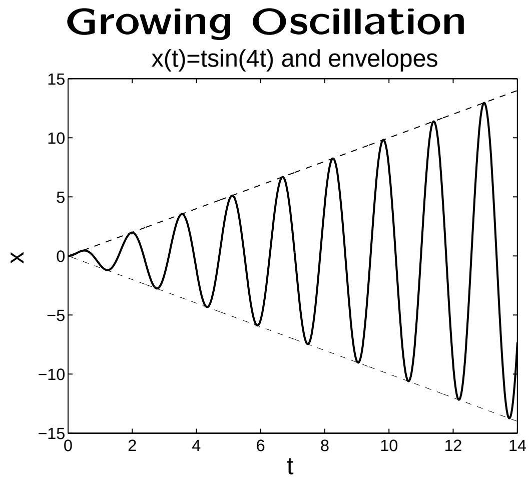
Solution (2) is not valid if $\omega = \omega_0$. In this case try

$$x_p(t) = t(a\cos\omega_0 t + b\sin\omega_0 t)$$

$$\Rightarrow x_p'' + \omega_0^2 x_p = -2a\sin\omega_0 t + 2b\cos\omega_0 t$$

The r.h.s. equals $A\cos\omega_0 t$ if $a = 0, 2\omega_0 b = A \Rightarrow b = A/(2\omega_0)$





$$\Rightarrow x_p(t) = [A/(2\omega_0)]t\sin\omega_0 t \quad \text{(linearly growing oscillation)}$$

Note: $x_p(0) = 0$, $x_p'(0) = 0$. **Ex.:** $A = 8$, $\omega_0 = 4 \Rightarrow x_p(t) = t\sin 4t$

(B) Damped Case

$$x'' + dx' + \omega_0^2 x = A\cos\omega t \qquad (6)$$

Since $A\cos\omega t = \mathrm{Re}(Ae^{i\omega t})$, any solution $x(t)$ is the real part of a solution $z(t)$ of

$$z'' + dz' + \omega_0^2 z = Ae^{i\omega t} \qquad (7)$$

Solution Strategy:

- Find particular solution of (7)
- Real part $\rightarrow$ particular solution of (6)

Try **complex exponential** for (7):

$$z_p(t) = ae^{i\omega t} \;\Rightarrow\; z_p'' + dz_p' + \omega_0^2 z_p = ((i\omega)^2 + i\omega d + \omega_0^2)ae^{i\omega t} = Ae^{i\omega t}$$

$$\Rightarrow [(\omega_0^2 - \omega^2) + i\omega d]a = A$$

$$\Rightarrow \frac{a}{A} = \frac{1}{(\omega_0^2 - \omega^2) + i\omega d}$$

Use $\quad 1/(\alpha + i\beta) = (\alpha - i\beta)/(\alpha^2 + \beta^2)$

$$\Rightarrow \frac{a}{A} = \frac{(\omega_0^2 - \omega^2) - i\omega d}{D}$$

where $D = (\omega_0^2 - \omega^2)^2 + \omega^2 d^2$

Amplitude and Phase: Set

$$a/A = Ge^{-i\phi} = G\cos\phi - iG\sin\phi$$

$$\Rightarrow G^2 = \left(\frac{(\omega_0^2-\omega^2)}{D}\right)^2 + \left(\frac{\omega d}{D}\right)^2$$
$$= \frac{(\omega_0^2-\omega^2)^2+\omega^2 d^2}{D^2} = \frac{D}{D^2}$$

$\Rightarrow G = 1/\sqrt{D} \equiv G(\omega)$ (*gain*), hence

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2 d^2}} \qquad (8)$$

Phase angle:

$$\omega_0^2-\omega^2 = G\cos\phi,\ \omega d = G\sin\phi$$

where $0 \le \phi < \pi$ (since $\sin\phi \ge 0$)

$$\Rightarrow \phi(\omega) = \text{arccot}\left(\frac{\omega_0^2-\omega^2}{\omega d}\right) \qquad (9)$$

Particular Solution of (6):

$$z_p(t) = ae^{i\omega t} = G(\omega)Ae^{i(\omega t-\phi)} \Rightarrow$$

$$x_p(t) = \text{Re}z_p(t) = GA\cos(\omega t-\phi) \quad (10)$$

General Solution of (6):

$$x(t) = x_h(t) + x_p(t) \qquad (11)$$

where $x_h(t) = c_1x_1(t) + c_2x_2(t)$ (12)

and $x_1(t)$, $x_2(t)$ is F.S.S. of

$$x'' + dx' + \omega_0^2 x = 0$$

Steady State and Transient Parts:

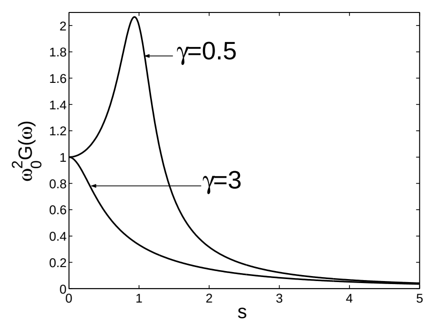
- $x_p(t)$: steady state part (persistent oscillation)
- $x_h(t)$: transient part ($d > 0$ $\Rightarrow x_h(t) \to 0$ for $t \to \infty$)

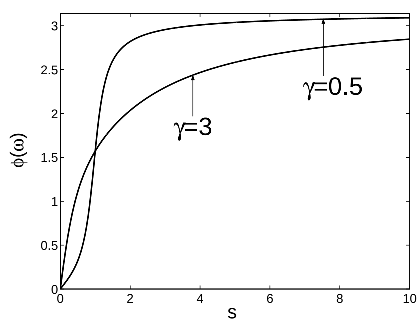
Qualitative Forms of $G(\omega)$, $\phi(\omega)$:

Set $s = \omega/\omega_0$, $\gamma = d/\omega_0 \Rightarrow$

$$\omega_0^2 G(\omega) = \frac{1}{\sqrt{(1-s^2)^2+s^2\gamma^2}}$$

$$\phi(\omega) = \text{arccot}\left(\frac{1-s^2}{s\gamma}\right)$$





- $G(\omega)$ has max at $s_m = \sqrt{1-\gamma^2/2}$, $\omega_0^2 G_m = 2/(\gamma\sqrt{4-\gamma^2})$, if $\gamma < \sqrt{2}$, and is monotonic for $\gamma > \sqrt{2}$
- $\phi(\omega)$ is "steep" for small γ and "flat" for large γ

Ex.: Consider a mass-spring system with $m = 5\,kg$, $\mu = 7\,kg/s$, $k = 3\,kg/s^2$, and a forcing term $2\cos 4t\,N$

(a) Find the steady periodic solution $x_p(t)$ and determine its amplitude and phase.

Answer: Equation: $5x'' + 7x' + 3x = 2\cos 4t \Rightarrow x'' + 1.4x' + 0.6x = 0.4\cos 4t$

Use complex method: $x_p(t) = \text{Re}z_p(t)$, where z_p is particular solution of

$$z'' + 1.4z' + 0.6z = 0.4e^{4it}$$

Try $z_p = ae^{4it} \Rightarrow (-16 + 5.6i + 0.6)ae^{4it} = 0.4e^{4it}$

$$\Rightarrow a = \frac{0.4}{-15.4 + 5.6i} = \frac{0.4 \times (-15.4 - 5.6i)}{15.4^2 + 5.6^2} = -0.0229 - 0.0083i$$

$$\begin{aligned} \Rightarrow z_p(t) &= (-0.0229 - 0.0083i)(\cos 4t + i\sin 4t) \\ \Rightarrow x_p(t) &= \text{Re}(z_p(t)) = 0.0083\sin 4t - 0.0229\cos 4t \text{ (superposition form)} \end{aligned}$$

To find amplitude and phase compute polar form: $a = A_0 e^{-i\phi}$, where

$$\begin{aligned} A_0 &= \sqrt{0.0229^2 + 0.0083^2} = 0.0244 \\ \phi &= \text{arccot}(-0.0229/0.0083) = 2.7939 \end{aligned}$$

$$\begin{aligned} \Rightarrow z_p(t) &= A_0 e^{i(4t-\phi)} \\ \Rightarrow x_p(t) &= 0.0244\cos(4t - 2.7939) \text{ (amplitude-phase form)} \end{aligned}$$

(b) Find the position $x(t)$ if $x(0) = 0$, $x'(0) = 1\, m/s$

Answer: Find transient part: $x'' + 1.4x' + 0.6x = 0 \Rightarrow p(\lambda) = \lambda^2 + 1.4\lambda + 0.6 = 0$
$\Rightarrow \lambda = -0.7 \pm 0.3317i$

$\Rightarrow x_h(t) = e^{-0.7t}[c_1 \cos(0.3317t) + c_2 \sin(0.3317t)]$ and $x(t) = x_h(t) + x_p(t)$

Match c_1, c_2 to IC: (use superposition form)

$$\left.\begin{aligned} x(0) &= c_1 - 0.0229 = 0 \Rightarrow c_1 = 0.0229 \\ x'(0) &= -0.7c_1 + 0.3317c_2 + 4 \times 0.0083 = 1 \Rightarrow c_2 = 2.9630 \end{aligned}\right\} \Rightarrow$$

$x(t) = e^{-0.7t}[0.0229 \cos(0.3317t) + 2.9630 \sin(0.3317t)] + 0.0244 \cos(4t - 2.7939)$

(c) Plot $x_p(t)$ and $x(t)$