4.5-6: Inhomogeneous Higher Order Equations
4.5: Method of Undetermined Coefficients

Ex.: $y'' + y = e^{-t}$

- **Trial Form:** $y_p(t) = ae^{-t}$
- Sub y_p in ODE $\Rightarrow$

$$y_p'' + y_p = ae^{-t} + ae^{-t} = 2ae^{-t} \stackrel{!}{=} e^{-t}$$

$\Rightarrow 2a = 1 \Rightarrow y_p(t) = e^{-t}/2$ is P.S.

Ex.: $y'' + y' + 2y = \cos t$

Try: sub $y_p = a\cos t$ in ODE $\Rightarrow$

$$\begin{aligned} & a(-\cos t - \sin t + 2\cos t) \\ = \ & a(\cos t - \sin t) \stackrel{?}{=} \cos t \end{aligned}$$

Doesn't work!
We need $\cos t$ *and* $\sin t$:

$$y_p(t) = a\cos t + b\sin t$$

$$\Rightarrow y_p'' + y_p' + 2y_p$$

$$= (a+b)\cos t + (-a+b)\sin t \stackrel{!}{=} \cos t$$

$$\Rightarrow \left\{ \begin{array}{rcl} a+b & = & 1 \\ -a+b & = & 0 \end{array} \right\} \Rightarrow a = b = 1/2$$

$\Rightarrow y_p(t) = (\cos t + \sin t)/2$ is P.S.

Ex.: $y'' + y = te^{-t}$

Try: sub $y_p(t) = ate^{-t}$ in ODE

$$\Rightarrow a(-2e^{-t} + te^{-t}) \stackrel{?}{=} te^{-t}$$

Doesn't work $\rightarrow$ use $y_p(t) = (a+bt)e^{-t}$

$$\begin{aligned} y_p'' + y_p &= [a + b(-2+t)]e^{-t} + (a+bt)e^{-t} \\ &= [(2a-2b) + 2bt]e^{-t} \stackrel{!}{=} te^{-t} \end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{rcl} 2a-2b & = & 0 \\ 2b & = & 1 \end{array} \right\} \Rightarrow a = b = 1/2$$

$\Rightarrow y_p(t) = (1+t)e^{-t}/2$ is P.S.

Ex.: $y'' - y = e^{-t}$

Try: sub $y_p(t) = ae^{-t}$ in ODE

$$\Rightarrow y_p'' - y_p = 0 \stackrel{?}{=} e^{-t}$$

Doesn't work $\rightarrow$ use $y_p(t) = ate^{-t}$

$$\begin{aligned} y_p'' - y_p &= a(-2+t)e^{-t} - a(te^{-t}) \\ &= -2ae^{-t} \stackrel{!}{=} e^{-t} \end{aligned}$$

$\Rightarrow a = -1/2 \Rightarrow y_p(t) = -te^{-t}/2$ is P.S.

General Method

Inhomogeneous nth Order ODE:

($a_j =$ const, $a_0 \neq 0$, $F(t)$: given forcing function)

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y(t) = F(t) \tag{1}$$

Assume $F(t)$ has the form

$$F(t) = P(t) e^{rt} \tag{2}$$

$$\text{or } F(t) = e^{rt}[P(t)\cos(\omega t) + Q(t)\sin(\omega t)] \tag{3}$$

where $P(t)$ or $P(t), Q(t)$ are polynomials in t

Let $M =$ degree of P in case of (2) ($M = 0$ if $P =$ const)

or $M =$ max of degrees of P, Q in case of (3)

In case of (2) set

$$\hat{y}_p(t) = p(t) e^{rt} \tag{4}$$

In case of (3) set

$$\hat{y}_p(t) = e^{rt}[p(t)\cos(\omega t) + q(t)\sin(\omega t)] \tag{5}$$

where $\left\{ \begin{array}{rcl} p(t) & = & p_0 + p_1 t + p_2 t^2 + \cdots p_M t^M \\ q(t) & = & q_0 + q_1 t + q_2 t^2 + \cdots q_M t^M \end{array} \right\}$

with undetermined coefficients p_0, p_1 etc.

Consider **characteristic equation** of homogeneous ODE:

$$p_c(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots a_{n-1}\lambda + a_n$$

Construct **trial function** $y_p(t)$ for (1) as follows:

- Let $\lambda_0 = \left\{ \begin{array}{cl} r & \text{for case of equation (2)} \\ r + i\omega & \text{for case of equation (3)} \end{array} \right\}$
  - If $p_c(\lambda_0) \neq 0$: Set $y_p(t) = \widehat{y}_p(t)$
  - If λ_0 is a root of $p_c(\lambda)$ of multiplicity m, i.e. $t^{m-1}e^{\lambda_0 t}$ is a solution of the homogenous ODE: Set $y_p(t) = t^m \widehat{y}_p(t)$

- Sub $y_p(t)$ in (1)
- Equate coefficients of all linearly independent terms
  $\Rightarrow$ system of linear equations for $p_0, \ldots, p_M$ [case of (2)] or $p_0, \ldots, p_M, q_0, \ldots, q_M$ [case of (3)] with unique solution

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Combinations of forcing terms

If $F(t)$ is a superposition of different functions of the forms (2) or (3) (different values of r or (r, ω)), treat each of them separately and add the resulting y_p's (Linearity!)

Complex Method for Trig Functions

Example:

$$y'' + 2y' - 3y = 5\sin 3t \qquad (6)$$

Consider complex solution of

$$z'' + 2z' - 3z = 5e^{3it} \qquad (7)$$

and note that

$$e^{3it} = \cos 3t + i\sin 3t$$

$\Rightarrow$ If $z_p(t)$ is P.S. of (7)
then $y_p(t) = \mathrm{Im} z_p(t)$ is P.S. of (6)

Trial form for (7): $z_p(t) = ae^{3it}$

Sub $z_p(t)$ in (7):

$$\begin{aligned}
& z_p'' + 2z_p' - 3z_p \\
= \; & a[(3i)^2 + 2(3i) - 3]e^{3it} \\
= \; & a(-9 + 6i - 3)e^{3it} \\
= \; & -a(12 - 6i)e^{3it} \\
= \; & -6a(2 - i)e^{3it} \\
\overset{!}{=} \; & 5e^{3it}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow a &= -\frac{5}{6}\frac{1}{2-i} \\
&= -\frac{5}{6}\frac{1}{2-i}\frac{2+i}{2+i} \\
&= -\frac{5}{6}\frac{2+i}{5} \\
&= -\frac{1}{6}(2+i)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow z_p(t) &= -\frac{1}{6}(2+i)e^{3it} \\
&= -\frac{1}{6}(2+i)(\cos 3t + i\sin 3t) \\
&= -\frac{1}{6}[(2\cos 3t - \sin 3t) \\
&\quad + i(\cos 3t + 2\sin 3t)]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow y_p(t) &= \mathrm{Im} z_p(t) \\
&= -\frac{1}{6}(\cos 3t + 2\sin 3t)
\end{aligned}$$

is P.S. of (6)

4.6: Variation of Parameters for 2nd Order ODEs

Inhomogeneous equation:

$$y'' + a(t)y' + b(t)y = F(t) \quad (8)$$

Homogeneous equation:

$$y'' + a(t)y' + b(t)y = 0 \quad (9)$$

Inhomogeneous system:

$$\mathbf{x}' = A(t)\mathbf{x} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \quad (10)$$

where $A(t) = \begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix}$

Let $y_1(t), y_2(t)$ be F.S.S for (9)

$$\Rightarrow X(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$

is F.M. of $\mathbf{x}' = A(t)\mathbf{x}$, and

$$X(t)^{-1} = \frac{1}{W(t)} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix}$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$.

Set $\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \int X(t)^{-1} \begin{bmatrix} 0 \\ F(t) \end{bmatrix} dt$

$$\Rightarrow \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \int \begin{bmatrix} -y_2(t) \\ y_1(t) \end{bmatrix} \frac{F(t)}{W(t)} dt$$

Particular solution of (10):

$$\begin{aligned} \mathbf{x}_p(t) &= X(t) \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \\ &= \begin{bmatrix} y_1(t)v_1(t) + y_2(t)v_2(t) \\ y_1'(t)v_1(t) + y_2'(t)v_2(t) \end{bmatrix} \end{aligned}$$

First component is part. sol. of (8):

$$y_p(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$$

where $\begin{cases} v_1(t) = -\int [y_2(t)F(t)/W(t)]\, dt \\ v_2(t) = \int [y_1(t)F(t)/W(t)]\, dt \end{cases}$

Ex.: Find particular solution of

$$y'' + y = \tan t$$

F.S.S. of $y'' + y = 0$:

$$\left.\begin{aligned} y_1(t) &= \cos t \\ y_2(t) &= \sin t \end{aligned}\right\} W(t) = \cos^2 t + \sin^2 t = 1$$

$$\begin{aligned} v_1(t) &= -\int \sin t \tan t\, dt \\ &= \sin t - \ln|\sec t + \tan t| \\ v_2(t) &= \int \cos t \tan t\, dt = \cos t \end{aligned}$$

$\Rightarrow y_p(t) = -\cos t \ln|\sec t + \tan t|$