Chapter 3: Modelling and Applications

Principle: Develop model function $f(t, x)$ for the rate of change of a variable $x \Rightarrow$ First order ODE: $x' = f(t, x)$

3.3: Personal Finance (Savings Accounts and Loans)

- $\mathbf{P(t)}$: balance on a savings account or loan; unit: \$; time measured in years
- $\mathbf{r}$: annual interest rate (5% $\rightarrow r = 0.05$/yr)
- $\mathbf{Q(t)}$: deposit/withdrawl (for savings accounts) or payment (loans) per year; unit: \$/yr

Continuous interest compound

$$\Rightarrow \quad P(t + \Delta t) - P(t) \approx rP(t)\Delta t + Q(t)\Delta t$$

or
$$\frac{dP}{dt} = rP + Q(t) \qquad (1)$$

Solution

Use variation of parameter:

$$P(t) = CP_h(t) + P_h(t) \int [Q(t)/P_h(t)]dt$$

$$P_h' = rP_h \;\Rightarrow\; P_h(t) = e^{rt}$$

$\Rightarrow$ (using definite integral)

$$P(t) = P_0 e^{rt} + e^{rt} \int_0^t e^{-r\tau} Q(\tau)\, d\tau \qquad (2)$$

where $C = P(0) \equiv P_0$

$$\text{If } Q(t) = Q_0 = \text{const}: \quad P(t) = e^{rt}(P_0 + Q_0/r) - Q_0/r \tag{3}$$

$$\textbf{Savings accounts:} \left\{ \begin{array}{ll} Q_0 = D > 0 & \text{if} \quad \text{deposit} \\ Q_0 = -W < 0 & \text{if} \quad \text{withdrawl} \end{array} \right\}$$

Loans: $Q_0 = -W < 0$; W: yearly payment

Examples for (3)

Example 3.4 Savings account without deposit/withdrawl: $P(t) = P_0 e^{rt}$

Assume $P_0 = \$\,1000$, $r = 0.05$.

After 40 *years:* $P(40) = 1000\, e^{0.05 \times 40} = 1000\, e^2 = \$\,7389$

Ex. 4: Investment in education. Assume $P_0 = 0$, $r = 0.0625$ (6.25 %).

Goal: $P(18) = \$\,50,000$. *Q:* What is D ?

$$P(t) = (D/r)(e^{rt} - 1) \rightarrow \text{solve } P(18) = (D/r)(e^{18r} - 1) = 50,000 \text{ for } D$$

$$\Rightarrow D = (50,000 \times 0.0625)/(e^{0.0625 \times 18} - 1) = \$\,1502.25$$

Ex. 6: (Loan). Assume $r = 0.08$, term $= 4\,$yrs, monthly payment $\$\,225$
$\Rightarrow W = \$\,12 \times 225/\text{yr} = \$\,2700/\text{yr}$. *Question:* Amount P_0 of loan?

$$\text{Solution: } P(t) = e^{rt}(P_0 - W/r) + W/r$$

$$\text{Solve } P(4) = e^{4r}(P_0 - W/r) + W/r = 0 \text{ for } P_0$$

$$\Rightarrow P_0 = (W/r)(1 - e^{-4r}) = (2700/0.08)(1 - e^{-4\times 0.08}) = \$\,9,242.45$$

Example 3.9 Saving for retirement.
Assume Savings account with $P_0 = 0$, $r = 0.05$ and annual deposit $D = \$\,2000$.
Q: retirement start up money $P(30) = ?$

$$\text{Solution: } P(t) = (D/r)(e^{rt} - 1)$$

After 30 years: $P(30) = (2000/0.05)(e^{0.05\times 30} - 1) = \$\,139,267.6$ (not much!)

Example 3.11 Computing start up money *needed* for retirement.
Given $r = 0.05$, annual withdrawl $W = \$\,50,000/\text{yr}$, find P_0 s.t. $P(30) = 0$.

$$\text{Solution: } P(t) = e^{rt}(P_0 - W/r) + W/r$$

$$\text{Solve } P(30) = e^{30r}(P_0 - W/r) + W/r = 0 \text{ for } P_0$$

$$\begin{aligned} \Rightarrow P_0 &= (W/r)(1 - e^{-30r}) = (50,000/0.05)(1 - e^{-30\times 0.05}) \\ &= 10^6(1 - e^{-1.5}) = \$\,770,896.8 \end{aligned}$$

Example for (2)

Example 3.12 Saving for retirement start up money *needed*. Model:

- $S(t)$: annual salary, S_0: initial salary, q: annual salary increase rate
$$\Rightarrow S' = qS \Rightarrow S(t) = S_0 e^{qt}$$
- ρ: fraction of annual salary deposited $\Rightarrow$ annual deposit:
$$D(t) = \rho S(t) = \rho S_0 e^{qt}$$
- r: interest rate on savings account, P_0: initial deposit.
Use Eq. (2) with $Q(t) = D(t) \Rightarrow$
$$\begin{aligned} P(t) &= P_0 e^{rt} + e^{rt}\int_0^t e^{-r\tau} D(\tau)d\tau = P_0 e^{rt} + \rho S_0 e^{rt}\int_0^t e^{-r\tau}e^{q\tau}d\tau \\ &= P_0 e^{rt} + \rho S_0 e^{rt}\int_0^t e^{(q-r)\tau}d\tau = P_0 e^{rt} + \rho S_0 e^{rt}[1/(r-q)](1-e^{(q-r)t}) \\ &= P_0 e^{rt} + [\rho S_0/(r-q)](e^{rt}-e^{qt}) \ \ (\text{if } q \neq r) \end{aligned}$$

Assume $P_0 = 0$. Given P^* and $t = h$, determine ρ such that $P(h) = P^* \Rightarrow$
$$[\rho S_0/(r-q)](e^{rh}-e^{qh}) = P^* \Rightarrow \rho = \mathbf{(r-q)P^*/[S_0(e^{rh}-e^{qh})]}$$

Number Example:
$r = 0.05$, $q = 0.04$, $h = 40\,\text{yrs}$, $P^* = \$\,1{,}600{,}000$, $S_0 = \$\,35{,}000$
$$\begin{aligned} \Rightarrow \rho &= 0.01 \times 1{,}600{,}000/[35{,}000(e^{40\times 0.05} - e^{40\times 0.04})] \\ &= (16/35)[1/(e^2 - e^{1.6})] = 0.19 = 19\,\% \end{aligned}$$

Hint for Ex. 10: Use $D(t) = 1000 + 500\,t$