Chapter 3: Modelling and Applications

Principle: Develop model function $f(t, x)$ for the rate of change of a variable $x \Rightarrow$ First order ODE: $x' = f(t, x)$

3.1: Modelling Population Growth

- $P(t)$: Population of species (bacteria, US-pop., ...)
- Model: $\frac{dP/dt}{P} = f(P)$
- Malthusian model:

  $f(P) = r = b - d =$ const

  b: birth rate, d: death rate

  $$\Rightarrow \quad \frac{dP}{dt} = rP$$
- Solution:

  $P(t) = P_0 e^{rt}$, $P_0 = P(0)$

- $\Rightarrow rt = \ln[P(t)/P_0]$

  Use this to determine
  - r if P_0, $P(t_1) = P_1$ are given
  - t^* if r, P_0, P^* are given and t^* is sought s.t. $P(t^*) = P^*$

Ex.: $r = 0.1$, $P_0 = 10^3 \rightarrow$

$$\begin{aligned} P(10) &= 2.7 \times 10^3 \\ P(100) &= 22 \times 10^6 \end{aligned}$$

(rapid growth → see text p. 125)

Examples

Ex.: At $t = 0$: $P_0 = 10$ cells. After 1 day: $P(1) = 25$ cells
Q: number of cells after 10 days?

$$r = (1/1)\ln(25/10) = 0.9163/\text{day} \;\Rightarrow\; P(10) = 10e^{10\times 0.9613} \approx 95.4\,\text{cells}$$

Ex. 2: A cell culture is grown at $t = 0$.
After $t_1 = 1$ day: $P_1 = 1000$. After $t_2 = 2$ days: $P_2 = 3000$.
Q: $P(0) =?$

$$r(t_2 - t_1) = \ln(P_2/P_1) \;\Rightarrow\; r = (1/1)\ln(3000/1000) = 1.099/\text{day}$$

$$\Rightarrow\; P_0 = P(1)e^{-r\times 1} = 1000e^{-1.099} \approx 333$$

Ex. 4: *Doubling Time:*
Given t_d s.t. $P(t_d) = 2P_0 \Rightarrow P_0 e^{rt_d} = 2P_0 \Rightarrow rt_d = \ln 2 \Rightarrow r = (\ln 2)/t_d$
Q: Given $t_d = 10$ days and $P_0 = 1000$, find t^* s.t. $P(t^*) = 10,000 \equiv P^*$

$$t_d = 10\,\text{days} \;\Rightarrow\; r = (\ln 2)/10 = 0.0693/\text{day}$$

$$\Rightarrow\; t^* = (1/r)\ln(P^*/P_0) = (\ln 10)/0.0693 \approx 33\,\text{days}$$

Logistic ODE: Avoidance of unlimited growth

Model: $(dP/dt)/P = r - aP$

- Set $K = r/a \Rightarrow$

  $$\frac{dP}{dt} = rP(1 - P/K) \equiv f(P) \quad (1)$$

- Equilibria: $P' = 0 \Rightarrow$

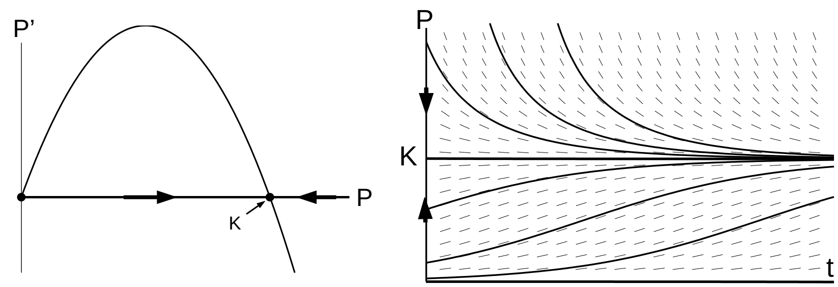
  $P = 0$: $f'(0) = r > 0$

  $\Rightarrow$ unstable

  $P = K$: $f'(K) = -r < 0$

  $\Rightarrow$ asympt. stable

Qualitative Analysis:



K: carrying capacity or eventual population

Solution of (1):

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}} \quad (2)$$

Derivation of (2). S.o.V.: $dP/[P(1 - P/K)] = [1/P - 1/(P - K)]dP = r\,dt$
$\Rightarrow \ln|P| - \ln|K - P| = \ln|P/(K - P)| = rt + C \Rightarrow P/(K - P) = Ae^{rt}$
For $t = 0$: $P_0/(K - P_0) = A \Rightarrow P_0(K - P)/[P(K - P_0)] = e^{-rt} \Rightarrow$ (2)

Computing Parameters:

- If K, P_0, $t = h$, $P_1 = P(h)$ are known:

$$P_1 = \frac{KP_0}{P_0 + (K - P_0)e^{-rh}}$$
$$\Rightarrow r = \frac{1}{h}\ln(\frac{P_1(K - P_0)}{P_0(K - P_1)})$$

- If P_0, $t = h$, $P_1 = P(h)$, $P_2 = P(2h)$ are known:

$$r = \frac{1}{h}\ln(\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)})$$
$$K = \frac{P_0P_1(1 - e^{-rh})}{P_0 - P_1e^{-rh}}$$
$$= \frac{P_1P_2(1 - e^{-rh})}{P_1 - P_2e^{-rh}}$$

Ex. 12: Given $K = 20,000$, $P_0 = 1000$ and $P_1 = P(8\,\text{hrs}) = 1200$, find r, and t^* s.t. $P(t^*) = 3K/4 = 15,000$.

$$r = (1/8)\ln(\frac{1.2(20-1)10^6}{1(20-1.2)10^6})$$
$$\approx 0.0241/\text{hr}$$

$$P^* = KP_0/[P_0 + (K - P_0)e^{-rt^*}]$$
$$\Rightarrow t^* = (1/r)\ln(\frac{P^*(K - P_0)}{P_0(K - P^*)})$$
$$= \frac{1}{0.0241}\ln(\frac{15(20-1)10^6}{1(20-15)10^6]})$$
$$\approx 72.22\,\text{hrs}$$

Ex. 14 (modified): Given $P_0 = 100$, $P_1 = P(20\,\text{hrs}) = 476$, $P_2 = P(40\,\text{hrs}) = 1986$, find r and K.

$$r = \frac{1}{20}\ln(\frac{1986(476-100)}{100(1986-476)})$$
$$\approx 0.0799$$
$$\Rightarrow K = \frac{476 \cdot 100(1 - e^{-0.08\cdot 20})}{100 - 476e^{-0.08\cdot 20}}$$
$$\approx 10,136$$

Effect of Harvesting:

Constant harvesting rate $H \Rightarrow$

$$P' = rP(1 - P/K) - H$$

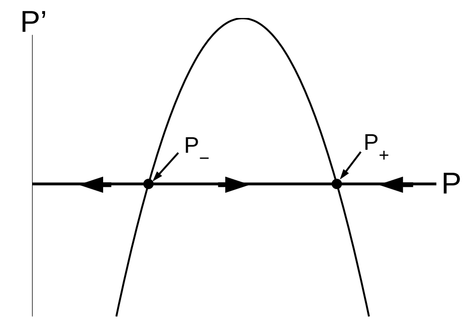
Equilibria:

$$P^2 - KP + HK/r = 0 \ \Rightarrow$$

$$P_\pm = (K/2)[1 \pm \sqrt{1 - 4H/(rK)}\,]$$

Note: $P_\pm$ exist if $H < rK/4$!

Qualitative Analysis:



- If $P_0 < P_- \Rightarrow$ population dies out
- If $P_0 > P_- \Rightarrow$ $P(t) \to P_+$ for $t \to \infty$

Hints for Exercises:

Ex. 9:
- Start from $P' = rP(1 - P/K)$
- Differentiate this w.r.t. t
- Sub P' in resulting equation
- Conclude

Ex. 15:
- Harvesting Problem:
- Proceed as discussed above for given numbers