2.7: Existence and Uniqueness of Solutions

Basic Existence and Uniqueness Theorem (EUT):

Suppose $f(t, x)$ is defined and continuous, and has a continuous partial derivative $\partial f(t, x)/\partial x$ on a rectangle R in the tx–plane. Then, given any initial point (t_0, x_0) in R, the initial value problem

$$x' = f(t, x), \;\; x(t_0) = x_0$$

has a unique solution $x(t)$ defined in an interval containing t_0. Furthermore, the solution will be defined at least until the solution leaves R.

Interval of Existence:

Largest interval in which a solution of a first order ODE can be defined.

Ex.: $tx' = x + 3t^2 \Rightarrow x' = x/t + 3t$

- f and $\partial f/\partial x$ are defined and continuous for any (t, x) if $t \neq 0$
- General solution (use Sec. 2.6):

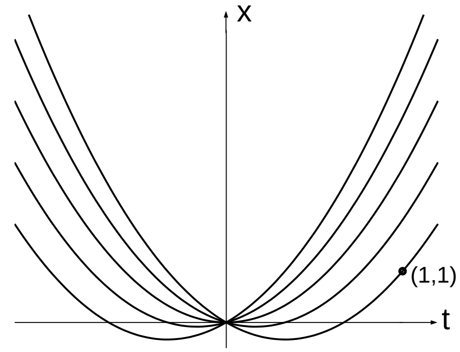
$$x(t) = 3t^2 + Ct$$

- For any C: $x(0) = 0$, hence
  - no solution for $x(0) = x_0 \neq 0$
  - ∞ solutions for $x(0) = 0$
- Solution for $x(t_0) = x_0$, $t_0 > 0$:

  $3t_0^2 + Ct_0 = x_0 \ \Rightarrow \ C = x_0/t_0 - 3t_0$

  $$\Rightarrow \ x(t) = 3t^2 + (x_0/t_0 - 3t_0)t$$

  unique solution with IoE $(0, \infty)$
- EUT applies to any rectangle that is not intersected by the vertical line $t = 0$.



Solution curves

$x(t) = 3t^2 + Ct$,

$C = -2, \ldots, 2$

Ex.: $x' = x^{1/3}$

S.o.V.: $\int x^{-1/3} dx = (3/2)x^{2/3} = t + D$

$\Rightarrow \ x_{\pm}(t) = \pm[(2/3)t + C]^{3/2} \ \ (C = 2D/3)$

- Let $C = 0 \Rightarrow x_{\pm}(0) = 0$
- Other solution with $x(0) = 0$:

$$x(t) = 0$$

$\Rightarrow$ At least 3 solutions for IC

$$x(0) = 0$$

- EUT doesn't apply to any rectangle that is intersected by the horizontal line $x = 0$

Ex.: $x' = -x^2, \;\; x(0) = x_0$

S.o.V.: $-\int (1/x^2)dx = 1/x = t + C$

$$\Rightarrow \; x = 1/(t+C)$$

$$x(0) = 1/C = x_0 \;\Rightarrow\; C = 1/x_0$$

$$\Rightarrow \; x(t) = x_0/(1 + x_0 t)$$

If $\left. \begin{array}{l} x_0 > 0 \\ x_0 < 0 \end{array} \right\} \Rightarrow$ IoE: $\left\{ \begin{array}{c} (-1/x_0, \infty) \\ (-\infty, -1/x_0) \end{array} \right.$

If $x_0 = 0 \;\Rightarrow\; x(t) = 0$, IoE: $(-\infty, \infty)$

- $f(t,x) = -x^2$ satisfies hypotheses of EUT in any rectangle

$\Rightarrow$ Unique solution for any x_0

- $x(t)$ leaves any rectangle in finite time

$\Rightarrow$ Solution is not defined for *all* reals if $x_0 \neq 0$

Ex.: IVP $y' = -2y + f(t), \;\; y(0) = 3$

$$f(t) = \left\{ \begin{array}{lll} 0 & \text{if} & t < 1 \\ 5 & \text{if} & t \geq 1 \end{array} \right.$$

$$t < 1 : \; y' = -2y \;\Rightarrow\; y(t) = 3e^{-2t}$$

For $t \to 1 : \; y(1) = 3e^{-2}$

Continue solution beyond $t = 1$:

$$t \geq 1 : \; y' = -2y + 5, \; y(1) = 3e^{-2}$$

$$\begin{aligned} \Rightarrow \; y(t) &= 3e^{-2t} + e^{-2t} \int_1^t e^{2t'} \, 5 \, dt' \\ &= 5/2 + (3 - 5e^2/2)e^{-2t} \end{aligned}$$

Combine:

$$y(t) = \left\{ \begin{array}{cll} 3e^{-2t} & \text{if} & t \leq 1 \\ 5/2 + (3 - 5e^2/2)e^{-2t} & \text{if} & t \geq 1 \end{array} \right.$$

- f is discontinuous at $t = 1$, but unique solution exists for all t
- $y'(t)$ is discontinuous at $t = 1$ (see text p.80 for graph)

Worked Out Examples from Exercises

Ex. 1: $y' = 4 + y^2$, $y(0) = 1$. Does IVP have a unique solution?

Yes, because $f = 4 + y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere.

Ex. 3: $y' = t \tan^{-1}(y)$, $y(0) = 2$. Does IVP have a unique solution?

Yes (as Ex. 1).

Ex. 5: $x' = t/(x+1)$, $x(0) = 0$. Does IVP have a unique solution?

Yes, because f and $\partial f/\partial x = -t/(x+1)^2$ are continuous in any rectangle away from the horizontal line $x = -1$, and $x(0) \neq -1$.

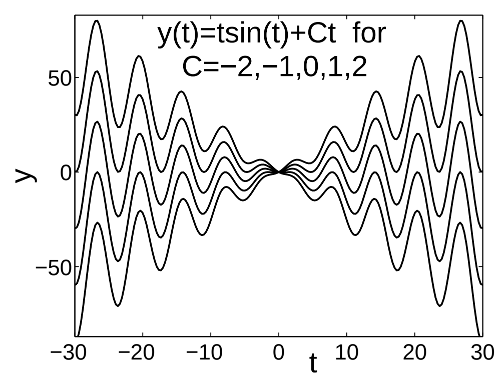
Ex. 7: $ty' - y = t^2 \cos t$, $y(0) = -3$.

(i) Find general solution and sketch several solutions.
(ii) Show IVP has no solution and explain why this doesn't contradict EUT.

Answer (i): $y' - y/t = t \cos t$, use integrating factor:

$$u(t) = \exp(-\int (1/t) dt) = \exp(-\ln t) = 1/t$$

$\Rightarrow (y/t)' = \cos t \;\Rightarrow\; y/t = \sin t + C \;\Rightarrow\; y(t) = t \sin t + Ct$



Answer (ii): Since $y(0) = 0$ for any C, there is no solution that satisfies $y(0) = -3$. This doesn't contradict EUT because f is not continuous at $t = 0$.