2.6: Exact Equations

Form: $\dfrac{dy}{dx} = -\dfrac{P(x,y)}{Q(x,y)}$

or $P(x,y) + Q(x,y)\dfrac{dy}{dx} = 0$ (1)

Differential form formulation:

$$P(x,y)dx + Q(x,y)dy = 0 \quad (2)$$

Level curves of a function F:

$$F(x,y) = C \quad (3)$$

On curves (3):

$$dF \equiv \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0 \quad (4)$$

- Curves (3) are **integral curves** for (1) if (3) defines implicit solutions of (1).
- (2) is **exact** if exists F s.t.

  $$P = \partial F/\partial x, \quad Q = \partial F/\partial y$$
- If (2) is exact, then (3) defines integral curves

Condition for exactness:

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial x \partial y}, \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

$$\Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad (5)$$

Find $\mathbf{F}$ (if (5) is satisfied):

- $\partial F/\partial x = P$

$\Rightarrow$ $F(x,y) = H(x,y) + \phi(y)$

$H(x,y) = \int P(x,y)\,dx$

- $\partial F/\partial y = Q$

$\Rightarrow$ $\frac{\partial}{\partial y} H(x,y) + \phi'(y) = Q(x,y)$

$\Rightarrow$ $\Phi'(y) = Q(x,y) - \frac{\partial}{\partial y} H(x,y)$

Solve this for $\Phi(y)$
(r.h.s. depends only on y)

Ex.: $e^y + (xe^y - \sin y)\frac{dy}{dx} = 0$

or $\quad e^y dx + (xe^y - \sin y)dy = 0$

$$\text{Here: } \begin{aligned} P(x,y) &= e^y \\ Q(x,y) &= xe^y - \sin y \end{aligned}$$

Check exactness:

$$\partial P/\partial y = e^y, \;\; \partial Q/\partial x = e^y$$

Equation is exact $\rightarrow$ **Find $\mathbf{F}$**

$$\begin{aligned} H(x,y) &= \int e^y dx = xe^y \\ \Phi'(y) &= Q - \partial H/\partial y \\ &= (xe^y - \sin y) - xe^y \\ &= -\sin y \\ \Rightarrow \;\; \Phi(y) &= \cos y \\ \Rightarrow \;\; F(x,y) &= H(x,y) + \Phi(y) \\ &= xe^y + \cos y \end{aligned}$$

Implicit Solutions:

$$xe^y + \cos y = C$$

Special Case: Separable Equations

$$\frac{dy}{dx} = -\frac{P(x)}{Q(y)}$$

$$P(x)dx + Q(y)dy = 0$$

$$\Rightarrow \quad \int P(x)\,dx + \int Q(y)\,dy = C$$

Ex.: $dy/dx = -x/y$

$$\Rightarrow \quad x + y\,dy/dx = 0$$

or $x\,dx + y\,dy = 0$

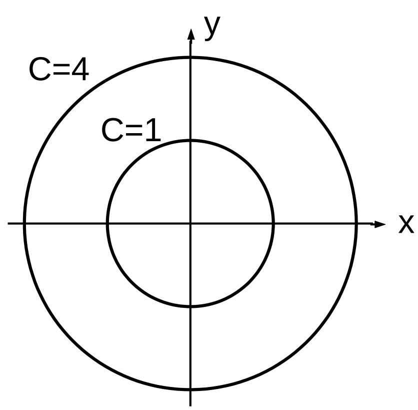
$$\int x\,dx + \int y\,dy = C$$

$$\Rightarrow \quad x^2/2 + y^2/2 = C$$

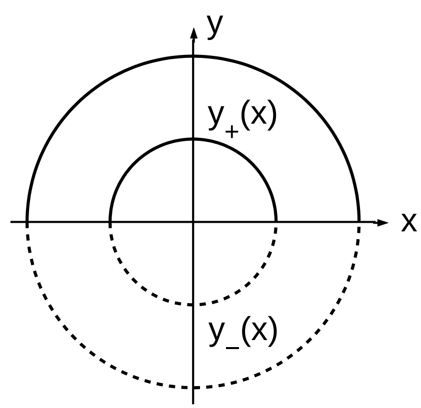
Integral curves are circles.

Explicit solutions:

$$y(x) = \pm\sqrt{2C - x^2}$$



Integral Curves



Explicit Solutions

Worked Out Examples from Exercises

(A) Calculate $\mathrm{dF} = (\partial \mathrm{F}/\partial \mathrm{x})\mathrm{dx} + (\partial \mathrm{F}/\partial \mathrm{y})\mathrm{dy}$ **for given** F

Ex. 1: $F(x,y) = 2xy + y^2 \;\Rightarrow\; dF = 2y\,dx + (2x + 2y)\,dy$

Ex. 3: $F(x,y) = \sqrt{x^2 + y^2} \;\Rightarrow\; dF = (1/\sqrt{x^2 + y^2})(x\,dx + y\,dy)$

Ex. 5: $F(x,y) = xy + \tan^{-1}(y/x)$

Use
$$\begin{aligned} \partial \tan^{-1}(y/x)/\partial x &= [1/(1 + y^2/x^2)](-y/x^2) = -y/(x^2 + y^2) \\ \partial \tan^{-1}(y/x)/\partial y &= [1/(1 + y^2/x^2)](1/x) = x/(x^2 + y^2) \end{aligned}$$

$$\begin{aligned} \Rightarrow\; dF &= [y - y/(x^2 + y^2)]dx + [x + x/(x^2 + y^2)]dy \\ &= [1/(x^2 + y^2)][(x^2 + y^2 - 1)y\,dx + (x^2 + y^2 + 1)x\,dy] \end{aligned}$$

Ex. 7: $F(x,y) = \ln(x^2 + y^2) + xy$

$$\Rightarrow\; dF = [y + 2x/(x^2 + y^2)]dx + [x + 2y/(x^2 + y^2)]dy$$

(B) Determine if equation is exact. If yes find implicit solution

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Ex. 9: $(2x+y)dx+(x-6y)dy=0$

$$P(x,y)=2x+y,\ \ Q(x,y)=x-6y \ \ \Rightarrow \ \ \partial P/\partial y=1,\ \partial Q/\partial x=1 \ \ \Rightarrow \ \ \text{exact}$$

$$H(x,y)=\int P(x,y)dx=x^2+xy \ \Rightarrow \ \Phi'(y)=Q-\partial H/\partial y=(x-6y)-x=-6y$$

$$\Rightarrow \ \Phi(y)=-3y^2 \ \Rightarrow \ F(x,y)=H(x,y)+\Phi(y)=x^2+xy-3y^2$$

$$\Rightarrow \text{implicit solution} \ \ x^2+xy-3y^2=C$$

Ex. 11: $(1+y/x)dx-(1/x)dy=0$

$$P=1+y/x,\ Q=-1/x \ \Rightarrow \ \partial P/\partial y=1/x,\ \partial Q/\partial x=1/x^2 \ \ \Rightarrow \ \ \text{not exact}$$

Ex. 13: $dy/dx=(3x^2+y)/(3y^2-x)$ or $(3x^2+y)dx+(x-3y^2)dy=0$

$$P=3x^2+y,\ Q=x-3y^2 \ \Rightarrow \ \partial P/\partial y=1,\ \ \partial Q/\partial x=1 \ \ \Rightarrow \ \ \text{exact}$$

$$H=\int P\,dx=x^3+xy \ \Rightarrow \ \Phi'=Q-\partial H/\partial y=(x-3y^2)-x=-3y^2 \ \Rightarrow \ \Phi=-y^3$$

$$\Rightarrow \ F=H+\Phi=x^3+xy-y^3 \ \Rightarrow \ \text{implicit solution} \ \ x^3+xy-y^3=C$$

Ex. 15: $(u+v)du + (u-v)dv = 0$

$$P(u,v) = u+v,\ Q(u,v) = u-v \ \Rightarrow\ \partial P/\partial v = 1,\ \partial Q/\partial u = 1 \ \Rightarrow\ \text{exact}$$

$$H = \int P\,du = u^2/2 + uv \ \Rightarrow\ \Phi'(v) = Q - \partial H/\partial v = (u-v) - u = -v$$

$$\Rightarrow\ \Phi(v) = -v^2/2 \ \Rightarrow\ F(u,v) = H(u,v) + \Phi(v) = u^2/2 + uv - v^2/2$$

$$\Rightarrow\ \text{implicit solution } u^2/2 + uv - v^2/2 = C$$

Ex. 17: $dr/ds = (\ln s)/(r/s - 2s)$ or $(r/s - 2s)dr - (\ln s)ds = 0$

$$P(r,s) = r/s - 2s,\ Q(r,s) = -\ln s \ \Rightarrow\ \partial P/\partial s = -r/s^2 - 2,\ \partial Q/\partial r = 0 \text{ (not exact)}$$

Ex. 19: $(\sin 2t)dx + (2x\cos 2t - 2t)dt = 0$

$$P = \sin 2t,\ Q = 2x\cos 2t - 2t; \Rightarrow\ \partial P/\partial t = 2\cos 2t,\ \partial Q/\partial x = 2\cos 2t \ \Rightarrow\ \text{exact}$$

$$H = \int P\,dx = x\sin 2t \ \Rightarrow\ \Phi'(t) = Q - \partial H/\partial t = (2x\cos 2t - 2t) - 2x\cos 2t = -2t$$

$$\Rightarrow\ \Phi = -t^2 \ \Rightarrow\ F = H + \Phi = x\sin 2t - t^2 \ \Rightarrow\ \text{implicit solution } x\sin 2t - t^2 = C$$

Ex. 21: $(2r + \ln y)dr + ry\,dy = 0$

$$P(r,y) = 2r + \ln y,\ Q(r,y) = ry \ \Rightarrow\ \partial P/\partial y = 1/y,\ \partial Q/\partial r = y \ \text{(not exact)}$$

Integrating Factors

If $P(x,y)\,dx + Q(x,y)\,dy = 0$

is not exact, try to find function $\mu(x,y)$ s.t.

$\mu P\,dx + \mu Q\,dy = 0$ is exact.

$\mu(\mathrm{x},\mathrm{y})$ = integrating factor

Ex.: $y(x-1)dx + x(y-1)dy = 0$

$$\left.\begin{array}{rcl} \partial[y(x-1)]/\partial y &=& x-1 \\ \partial[x(y-1)]/\partial x &=& y-1 \end{array}\right\} \text{not exact}$$

Multiply equation by $\mu = 1/(xy) \Rightarrow$

$$[(x-1)/x]dx + [(y-1)/y]dy = 0$$

$$\left.\begin{array}{rcl} \partial[(x-1)/x]/\partial y &=& 0 \\ \partial[(y-1)/y]/\partial x &=& 0 \end{array}\right\} \text{exact (separable)}$$

Direct integration $\Rightarrow$

$$F = x + y - \ln(xy)$$

Ex.: $(x+y)\,dx - x\,dy = 0$

$$\left.\begin{array}{rcl} \partial(x+y)/\partial y &=& 1 \\ \partial(-x)/\partial x &=& -1 \end{array}\right\} \text{not exact}$$

Use $\mu = 1/x^2 \Rightarrow$

$$(1/x + y/x^2)dx - (1/x)dy = 0$$

$$\left.\begin{array}{rcl} \partial(1/x + y/x^2)/\partial y &=& 1/x^2 \\ \partial(-1/x)/\partial x &=& 1/x^2 \end{array}\right\} \text{exact}$$

Find F:

$$H = \int (1/x + y/x^2)dx = \ln|x| - y/x$$

$$\Phi' = (-1/x) - \partial(\ln|x| - y/x)\partial y = 0$$

$$\Rightarrow\ \Phi = 0\ \Rightarrow\ F = H = \ln|x| - y/x$$

Homogeneous Equations

- A function $G(x, y)$ is **homogeneous of degree $\mathbf{n}$** if

$$G(\lambda x, \lambda y) = \lambda^n G(x, y)$$

- The DE $$P(x, y)\, dx + Q(x, y)\, dy = 0$$

  is homogeneous if P, Q are homogeneous of **same** degree.

- If the DE is homogeneous, then the substitution $y = xv$ transforms it to the separable DE:

$$\frac{dx}{x} + \frac{Q(1, v)dv}{P(1, v) + vQ(1, v)} = 0$$

Ex.: $G(x, y) = 1/(x^2 + y^2) \Rightarrow G(\lambda x, \lambda y) = 1/[(\lambda x)^2 + (\lambda y)^2] = \lambda^{-2}/(x^2 + y^2)$
$\Rightarrow G(\lambda x, \lambda y) = \lambda^{-2} G(x, y) \Rightarrow$ homogeneous of degree -2

Ex.: $G(x, y) = x^3 + xy^2 \Rightarrow G(\lambda x, \lambda y) = (\lambda x)^3 + (\lambda x)(\lambda y)^2 = \lambda^3(x^3 + xy^2)$
$\Rightarrow G(\lambda x, \lambda y) = \lambda^3 G(x, y) \Rightarrow$ homogeneous of degree 3

Ex.: $G(x, y) = x/y \Rightarrow G(\lambda x, \lambda y) = (\lambda x)/(\lambda y) = x/y = G(x, y) \Rightarrow$ degree 0

Worked Out Examples from Exercises

(C) Solution of Homogeneous Equations

Ex. 35: $(x^2+y^2)dx - 2xy\,dy = 0$

Sub $y = xv \;\Rightarrow\; dy = v\,dx + x\,dv \;\Rightarrow\; (x^2+x^2v^2)dx - 2x^2v(v\,dx + x\,dv) = 0$

$$\Rightarrow\; (1-v^2)dx - 2xv\,dv = 0 \;\Rightarrow\; dx/x + [2v/(v^2-1)]dv = 0$$

$$\Rightarrow\; \ln|x| + \ln|v^2-1| = \ln|x(v^2-1)| = D \;\Rightarrow\; x(v^2-1) = C \;\; (C = \pm e^D)$$

Sub $v = y/x \;\Rightarrow\; y^2 - x^2 = Cx$

Ex. 37: $(3x+y)dx + x\,dy = 0$

Sub $y = xv \;\Rightarrow\; dy = v\,dx + x\,dv \;\Rightarrow\; (3x+xv)dx + x(v\,dx + x\,dv) = 0$

$$\Rightarrow\; (3+2v)dx + x\,dv = 0 \;\Rightarrow\; (1/x)dx + [1/(3+2v)]dv = 0$$

$$\Rightarrow\; \ln|x| + (1/2)\ln|3+2v| = D \;\Rightarrow\; 2\ln|x| + \ln|3+2v| = 2D$$

$$\Rightarrow\; \ln(x^2|3+2v|) = 2D \;\Rightarrow\; x^2(3+2v) = C \;\; (C = \pm e^{2D})$$

Sub $v = y/x \;\Rightarrow\; 3x^2 + 2xy = C$