2.4: Linear Equations

General Form:

$$x' = a(t)x + f(t) \qquad (1)$$

If $f(t) = 0$, (1) is called **homogeneous**:

$$x' = a(t)x$$

If $f(t) \neq 0$, (1) is called **nonhomogeneous**

Examples of linear equations:

$x' = \sin(t)x$	homogenous, $a(t) = \sin t$
$x' = x/t$	homogenous, $a(t) = 1/t$
$y' = e^t y + \cos t$	nonhomogeneous, $a(t) = e^t$, $f(t) = \cos t$
$x' = 3tx + t^2$	nonhomogeneous, $a(t) = 3t$, $f(t) = t^2$

Examples of non-linear equations:

$$\begin{aligned} x' &= t \sin x \\ y' &= 1/y \\ y' &= 1 - y^2 \\ u' &= e^{-u} + \cos x \end{aligned}$$

Homogeneous Equation

HE: $x' = a(t)x \qquad (2)$

$$\Rightarrow \quad \frac{dx}{x} = a(t)dt$$

$$\Rightarrow \quad \ln|x| = \int a(t)dt + D$$

$$\Rightarrow \quad |x(t)| = \exp(D + \int a(t)dt)$$

$$= e^D \exp\Big(\int a(t)dt\Big)$$

$\Rightarrow$ **General Solution**:

$$x(t) = C\exp\Big(\int a(t)dt\Big) \qquad (3)$$

where $C = \pm e^D$ (any value)

Equivalent form of gen. sol.:

$$x(t) = x_0 \exp\Big(\int_{t_0}^{t} a(t')dt'\Big) \quad (4)$$

where $x_0 = x(t_0)$.

Ex.: $x' = \sin(t)x \quad (a(t) = \sin t)$

$$\Rightarrow \quad \int a(t)\,dt = \int \sin(t)\,dt = -\cos t$$

$$\Rightarrow \quad x(t) = Ce^{-\cos t}$$

Ex.: $x' = x/t \quad (a(t) = 1/t)$

$$\Rightarrow \quad \int a(t)\,dt = \int \frac{dt}{t} = \ln|t|$$

$$\Rightarrow \quad x(t) = Ce^{\ln|t|} = C|t| \quad (t \neq 0)$$

Since either $t > 0$ or $t < 0$:

$$\Rightarrow \quad x(t) = Bt \quad (t \neq 0,\ B = \pm C)$$

Nonhomogeneous Equation: Integrating Factor

NHE: $x' - a(t)x = f(t) \quad (5)$

Multiply by **integrating factor** $u(t)$ (determined below):

$$u(t)x' - u(t)a(t)x = u(t)f(t) \quad (6)$$

If $u(t)$ satisfies

$$u' = -a(t)u \quad (7)$$

then

$$ux' - uax = ux' + u'x = (ux)'$$

$$\Rightarrow \quad (u(t)x)' = u(t)f(t)$$

$$\Rightarrow \ u(t)x(t) = \int u(t)f(t)dt + C$$

$\Rightarrow$ **General Solution:**

$$x(t) = \frac{1}{u(t)} \int u(t)f(t)dt + C/u(t) \quad (8)$$

where $u(t)$ is a (part.) solution to the HE (7) (cf. (2) and (3)):

$$u(t) = \exp\Big(-\int a(t)dt\Big) \quad (9)$$

Ex.: $x' - x = e^{-t}$ $(a(t) = 1,\ f(t) = e^{-t})$

$$\int a(t)\,dt = \int dt = t \ \Rightarrow \ u(t) = e^{-t}$$

$$ux' - ux = e^{-t}x' - e^{-t}x = (e^{-t}x)',\ ue^{-t} = e^{-2t}$$

$$(e^{-t}x)' = e^{-2t} \ \Rightarrow \ e^{-t}x = \int e^{-2t}dt$$

$$e^{-t}x = -e^{-2t}/2 + C \ \Rightarrow \ x(t) = -e^{-t}/2 + Ce^{t}$$

Nonhomogeneous Equation: Variation of Parameter

$$\text{Set}\quad x_h(t) = 1/u(t) \qquad (10)$$
$$= \exp\Big(\int a(t)dt\Big)$$

and rewrite (8) as

$$x(t) = Cx_h(t) + x_p(t) \qquad (11)$$

where

$$x_p(t) = x_h(t)v(t) \qquad (12)$$

with

$$v(t) = \int \frac{f(t)}{x_h(t)}\,dt \qquad (13)$$

Eq (12) with (13) is called **Variation of Parameter Formula**

Terms in Gen. Sol. (11):

$Cx_h(t)$: Gen. Sol. of HE (2)

$x_p(t)$: Part. Sol. of NHE (5)

Ex.: $x' = x\tan t + \sin t$

HE: $x' = x\tan t \;\Rightarrow\; x_h(t) = \exp\Big(\int \tan t\,dt\Big)$

$\Rightarrow\; x_h(t) = \exp(\ln(1/\cos t)) = 1/\cos t$

$$v(t) = \int [f(t)/x_h(t)]\,dt = \int \sin t\cos t\,dt = -\cos^2 t/2$$

$$x_p(t) = x_h(t)v(t) = (1/\cos t)(-\cos^2 t/2) = -\cos t/2$$

$\Rightarrow$ Gen. Sol.: $x(t) = -\cos t/2 + C/\cos t$

Worked Out Examples from Exercises

(A) Find general solutions

Ex. 3: $y' + (2/x)y = (\cos x)/x^2$; use integrating factor

$$a(x) = -2/x \;\Rightarrow\; u(x) = \exp(-\int a(x)dx) = \exp(2\ln x) = \exp(\ln x^2) = x^2$$

Multiply ODE by $x^2 \Rightarrow x^2y' + 2xy = \cos x \;\Rightarrow\; (x^2y)' = \cos x \;\Rightarrow\; x^2y = \int \cos x\, dx$

$$\Rightarrow\; x^2y = \sin x + C \;\Rightarrow\; y(x) = (\sin x + C)/x^2$$

Ex. 5: $x' - 2x/(t+1) = (t+1)^2$; use variation of parameter

HE: $x' = 2x/(t+1) \;\Rightarrow\; x_h(t) = \exp(\int [2/(t+1)]dt) = \exp(\ln[(t+1)^2]) = (t+1)^2$

$$v(t) = \int [f(t)/x_h(t)]dt = \int dt = t \;\Rightarrow\; \text{part. sol.: } x_p(t) = x_h(t)v(t) = (t+1)^2 t$$

$$\text{gen. sol.: } x(t) = x_p(t) + Cx_h(t) = (t+1)^2(t+C)$$

Ex. 7: $(1+x)y' + y = \cos x \Rightarrow y' + y/(1+x) = (\cos x)/(1+x)$; use int. factor

$$u(x) = \exp(-\int a(x)dx) = \exp(\int [1/(1+x)]\,dx) = 1+x$$

Multiply ODE by $u(x) \ \Rightarrow \ (1+x)y' + y = [(1+x)y]' = \cos x$

$$\Rightarrow \ (1+x)y = \int \cos x\,dx = \sin x + C \ \Rightarrow \ y(x) = (\sin x + C)/(1+x)$$

Ex. 11: $y' = \cos x - y\sec x$; use variation of parameter

$$\text{HE: } y' = -y\sec x \ \Rightarrow \ y_h(x) = \exp(-\int \sec x\,dx)$$

$$\Rightarrow \ y_h(x) = \exp(-\ln(\sec x + \tan x)) = 1/(\sec x + \tan x)$$

$$\Rightarrow \ v(x) = \int [f(x)/y_h(x)]dx = \int (\sec x + \tan x)\cos x\,dx = \int (1+\sin x)dx = x - \cos x$$

$$\Rightarrow \ y_p(x) = y_h(x)v(x) = (x - \cos x)/(\sec x + \tan x)$$

$$\Rightarrow \ y(x) = y_p(x) + Cy_h(x) = (x - \cos x + C)/(\sec x + \tan x)$$

Ex. 12: $x' - (n/t)x = e^t t^n$; use integrating factor

$$u(t) = \exp(-\int (n/t)dt) = \exp(-n \ln t) = \exp(\ln t^{-n}) = t^{-n}$$

Multiply ODE by $u(t) \Rightarrow (t^{-n}x)' = e^t \Rightarrow (t^{-n}x) = \int e^t dt = e^t + C$

$$\Rightarrow \quad x(t) = t^n(e^t + C)$$

Example: $y' - ry = f(t)$; use variation of parameter

$$y_h(t) = \exp(\int r\,dt) = e^{rt}, \quad v(t) = \int [f(t)/y_h(t)]dt = \int e^{-rt} f(t)\,dt$$

$$\Rightarrow \text{ gen. sol.: } y(t) = e^{rt}(\int e^{-rt} f(t)\,dt + C)$$

If $f(t) = a = \text{const} \Rightarrow \int e^{-rt} f(t)\,dt = a \int e^{-rt} dt = -(a/r)e^{-rt} \Rightarrow y(t) = Ce^{rt} - a/r$

Ex. 33: $ty' + y = 4t^2$

Here one sees directly: $(ty)' = ty' + y = 4t^2 \Rightarrow ty = (4/3)t^3 + C$

$$\Rightarrow \quad y(t) = (4/3)t^2 + C/t$$

Ex. 35: $y' + 2xy = 4x$; use variation of parameter

$$y_h(x) = \exp\left(\int (-2x)dx\right) = e^{-x^2} \ \Rightarrow \ v(x) = \int 4xe^{x^2} dx = 2e^{x^2}$$

$$\Rightarrow \ y_p(x) = y_h(x)v(x) = 2 \ \Rightarrow \ y(x) = 2 + Ce^{-x^2}$$

(B) Solutions to IVPs

Ex. 15: $(x^2 + 1)y' + 3xy = 6x$, $y(0) = -1$; use integrating factor

normal form: $y' + [3x/(1+x^2)]y = 6x/(1+x^2) \ \Rightarrow \ u(x) = \exp\left(\int [3x/(1+x^2)]dx\right)$

$$\Rightarrow \ u(x) = \exp((3/2)\ln(1 + x^2)) = (1 + x^2)^{3/2} \ \Rightarrow \ (uy)' = 6x(1 + x^2)^{1/2}$$

$$\Rightarrow \ uy = \int 6x(1 + x^2)^{1/2} dx = 2(1 + x^2)^{3/2} + C \ \Rightarrow \ y(x) = 2 + C(1 + x^2)^{-3/2}$$

Invoke IC: $y(0) = 2 + C = -1 \ \Rightarrow \ C = -3 \ \Rightarrow \ y(x) = 2 - 3(1 + x^2)^{-3/2}$

Ex. 17: $x' + x\cos t = (1/2)\sin 2t = \sin t \cos t$, $x(0) = 1$; use variation of parameter

$$x_h(t) = \exp(-\int \cos t\, dt) = e^{-\sin t} \;\Rightarrow\; v(t) = \int \sin t \cos t e^{\sin t} dt = e^{\sin t}(\sin t - 1)$$

$$\Rightarrow\; x_p(t) = \sin t - 1 \;\Rightarrow\; \text{gen. sol.: } x(t) = \sin t - 1 + Ce^{-\sin t}$$

$$\text{Invoke IC: } x(0) = -1 + C = 1 \;\Rightarrow\; C = 2 \;\Rightarrow\; x(t) = \sin t - 1 + 2e^{-\sin t}$$

Ex. 19: $(2x+3)y' = y + (2x+3)^{1/2}$, $y(-1) = 0$; use integrating factor

$$\text{normal form: } y' = y/(2x+3) + (2x+3)^{-1/2} \;\Rightarrow\; u(x) = \exp(\int [-1/(2x+3)]dx)$$

$$\Rightarrow\; u(x) = \exp(-(1/2)\ln(2x+3)) = (2x+3)^{-1/2} \;\Rightarrow\; (uy)' = (2x+3)^{-1}$$

$$\Rightarrow\; uy = \int (2x+3)^{-1} dx = (1/2)\ln(2x+3) + C$$

$$\Rightarrow\; y(x) = (1/2)(2x+3)^{1/2}(\ln(2x+3) + C), \;\; \text{IC} \;\Rightarrow\; y(-1) = C/2 = 0$$

$$\Rightarrow\; y(x) = (1/2)(2x+3)^{1/2}\ln(2x+3), \;\; \text{IoE: } (-3/2, \infty)$$

Ex. 21: $(1+t)x' + x = \cos t$, $x(-\pi/2) = 0$

One sees directly: $[(1+t)x]' = (1+t)x' + x = \cos t \Rightarrow (1+t)x = \int \cos t\,dt$

$$\Rightarrow\ (1+t)x = \sin t + C\ \Rightarrow\ x(t) = (\sin t + C)/(1+t)$$

$$x(-\pi/2) = (-1+C)/(1-\pi/2) = 0\ \Rightarrow\ C = 1\ \Rightarrow\ x(t) = (1+\sin t)/(1+t),\ \text{IoE: } (-\infty, -1)$$

Ex. 37: $y' + y/2 = t$, $y(0) = 1$;

From Example p.7 ($r = -1/2$, $f(t) = t$): $y(t) = e^{-t/2}(\int e^{t/2}t\,dt + C)$

$$\int e^{t/2}t\,dt = 2(t-2)e^{t/2}\ \Rightarrow\ y(t) = 2(t-2) + Ce^{-t/2}$$

IC: $y(0) = -4 + C = 1\ \Rightarrow\ C = 5\ \Rightarrow\ y(t) = 2(t-2) + 5e^{-t/2}$

Ex. 39: $y' + 2xy = 2x^3$, $y(0) = -1$; use variation of parameter

$$y_h(x) = \exp\left(\int(-2x)dx\right) = e^{-x^2}\ \Rightarrow\ v(x) = \int 2x^3e^{x^2}dx = (x^2-1)e^{x^2}\ \Rightarrow\ y_p(x) = x^2-1$$

$\Rightarrow\ y(x) = x^2 - 1 + Ce^{-x^2}$, IC: $y(0) = C - 1 = -1\ \Rightarrow\ C = 0\ \Rightarrow\ y(x) = x^2 - 1$

Hint for Ex. 29: Use Newton's law of cooling with surrounding temperature $A(t) = -t$. Use a solver to solve the equation $T(t) = 37$ for t.