2.2: Solutions to Separable Equations

Form: $\frac{dy}{dt} = g(t)f(y)$

Implicit Solution:

$$[1/f(y)]dy = g(t)dt$$

$$\int [1/f(y)]dy = \int g(t)dt \quad (*)$$

or $H(y) = G(t) + C$ where

$$H(y) = \int [1/f(y)]dy$$

$$G(t) = \int g(t)dt$$

Definite integrals (for IVPs):

$$\int_{y_0}^{y} \frac{dy}{f(y)} = \int_{t_0}^{t} g(t)dt$$

Solve $(*)$ for $y \rightarrow$ explicit solution

Note: $(*)$ may have several solutions. Use IC to choose the right one.

Ex.: $\frac{dy}{dt} = ty^2$

$$(1/y^2)dy = t\,dt$$

$$\Rightarrow \quad \int (1/y^2)dy = \int t\,dt$$

$$\Rightarrow \quad -1/y = t^2/2 + C$$

$$\Rightarrow \quad y(t) = -1/(t^2/2 + C) = -2/(t^2 + 2C)$$

Ex.: Find gen. sol. to $dx/dt = rx$

$$\frac{dx}{x} = r\,dt \quad \Rightarrow \quad \ln|x| = rt + C$$

$$\Rightarrow \quad |x(t)| = e^{rt+C} = e^C e^{rt}$$

$$x(t) > 0 \quad \Rightarrow \quad x(t) = e^C e^{rt}$$
$$x(t) < 0 \quad \Rightarrow \quad x(t) = -e^C e^{rt}$$

Set $A = e^C$ if $x > 0$, $A = -e^C$ if $x < 0$

$$\Rightarrow \quad x(t) = Ae^{rt}$$

with arbitrary constant A (can be 0)
Initial value: $x(0) = A$

Ex.: Find sols. of $y' = e^x/(1+y)$ s.t. $y(0) = 1$ and $y(0) = -4$.

$$(1+y)dy = e^x dx \quad \Rightarrow \quad y + y^2/2 = e^x + C$$

$$\Rightarrow \quad y^2 + 2y - 2(e^x + C) = 0$$

$$\begin{aligned} y(x) &= \frac{1}{2}[-2 \pm \sqrt{4 + 8(e^x + C)}] \\ &= -1 \pm \sqrt{1 + 2(e^x + C)} \end{aligned}$$

(i) For $y(0) = 1 \Rightarrow$ need '+'–sign:

$$y(0) = -1 + \sqrt{3 + 2C} = 1 \quad \Rightarrow \quad C = 1/2$$

$$\Rightarrow \quad y(x) = -1 + \sqrt{2 + 2e^x}$$

(ii) For $y(0) = -4 \Rightarrow$ need '−'–sign:

$$y(0) = -1 - \sqrt{3 + 2C} = -4 \quad \Rightarrow \quad C = 3$$

$$\Rightarrow \quad y(x) = -1 - \sqrt{7 + 2e^x}$$

Worked out Examples from Exercises

(A) Find general solutions

Ex. 1: $y' = xy$

$$(1/y)dy = xdx \;\Rightarrow\; \ln|y| = x^2/2 + C \;\Rightarrow\; |y| = \exp(x^2/2 + C) = e^C e^{x^2/2}$$

$$\Rightarrow\; y(x) = Ae^{x^2/2}, \;\; A = e^C \text{ or } A = -e^C$$

Ex. 3: $y' = e^{x-y}$

$$e^y dy = e^x dx \;\Rightarrow\; e^y = e^x + C \;\Rightarrow\; y(x) = \ln(e^x + C)$$

Ex. 5: $y' = y(x+1)$

$$(1/y)dy = (x+1)dx \;\Rightarrow\; \ln|y| = x^2/2 + x + C \;\Rightarrow\; |y| = e^C e^{x+x^2/2} \;\Rightarrow\; y(x) = Ae^{x+x^2/2}$$

Ex. 9: $x^2 y' = y\ln y - y' \Rightarrow y' = (y\ln y)/(1+x^2)$

$$[1/(y\ln y)]dy = [1/(1+x^2)]dx \;\Rightarrow\; \ln(\ln y) = \arctan x + C$$

$$\Rightarrow\; y(x) = \exp(e^C e^{\arctan x}) = \exp(De^{\arctan x}) \;\; (D = e^C)$$

Ex. 11: $y^3 y' = x + 2y' \Rightarrow y' = x/(y^3 - 2)$

$(y^3-2)dy = x\,dx \;\Rightarrow\; y^4/4-2y = x^2/2+C \;\Rightarrow$ implicit sol.: $y^4-8y-2x^2 = D\;\; (D = 4C)$

(B) Find solutions to IVPs and IoEs

Ex. 13: $y' = y/x$, IC: $y(1) = -2$

General sol.: $(1/y)dy = (1/x)dx \Rightarrow \ln|y| = \ln|x| + C$

$\Rightarrow\; |y| = \exp(C + \ln|x|) = e^C e^{\ln|x|} = e^C|x| \;\Rightarrow\; y(x) = Ax\;\; (A = \pm e^C)$

Match C to IC: $y(1) = A = -2 \Rightarrow y(x) = -2x$; IoE: $(0, \infty)$

Ex. 15: $y' = (\sin x)/y$, IC: $y(\pi/2) = 1$

$y\,dy = \sin x\,dx \;\Rightarrow\; y^2/2 = -\cos x + C \;\Rightarrow\; y = \pm\sqrt{D - 2\cos x}\;\; (D = 2C)$

$y(\pi/2) = 1 > 0 \;\Rightarrow$ need $'+'$–sign $\;\Rightarrow\; y(\pi/2) = \sqrt{D} = 1 \;\Rightarrow\; y(x) = \sqrt{1 - 2\cos x}$

Find IoE: need $\cos x < 1/2 \Rightarrow$ IoE: $(\pi/3, 5\pi/3)$

Ex. 17: $y' = 1 + y^2$, IC: $y(0) = 1$

$$[1/(1+y^2)]dy = dt \;\Rightarrow\; \arctan y = t + C \;\Rightarrow\; y = \tan(t+C) + k\pi \;\; (k: \text{ integer})$$

$$\text{Since } y(0) = 1 \Rightarrow k = 0 \quad\Rightarrow\quad y(t) = \tan(t+C)$$

$$\text{Invoke IC: } y(0) = \tan C = 1 \Rightarrow C = \pi/4 \Rightarrow y(t) = \tan(t+\pi/4)$$

For IoE: need $t + \pi/4 > -\pi/2$ and $t + \pi/4 < \pi/2 \Rightarrow$ IoE: $(-3\pi/4, \pi/4)$

Ex. 19: $y' = x/y$, IC_1: $y(0) = 1$ and IC_2: $y(0) = -1$

$$y\,dy = x\,dx \;\Rightarrow\; y^2/2 = x^2/2 + C \;\Rightarrow\; y = \pm\sqrt{x^2 + D} \;\; (D = 2C)$$

$$\text{IC}_1\text{: } y(0) = 1 \;\Rightarrow\; y(0) = +\sqrt{D} = 1 \;\Rightarrow\; y(x) = \sqrt{1+x^2}$$

$$\text{IC}_2\text{: } y(0) = -1 \;\Rightarrow\; y(0) = -\sqrt{D} = -1 \;\Rightarrow\; y(x) = -\sqrt{1+x^2}$$

Example: general linear equation with constant coefficients

$y' = ry + a$, IC: $y(0) = y_0$ $\quad$ (r, a, y_0: arbitrary parameters)

$$[1/(ry+a)]dy = dt \;\Rightarrow\; (\ln|ry+a|)/r = t + C \;\Rightarrow\; |ry+a| = e^{rt+rC} = e^{rC}e^{rt}$$

$$\Rightarrow\; ry + a = Ae^{rt} \;\; (A = \pm e^{rC}) \;\Rightarrow\; y(t) = (Ae^{rt} - a)/r$$

$$\text{Invoke IC: } y(0) = (A-a)/r = y_0 \;\Rightarrow\; A = ry_0 + a \;\Rightarrow\; y(t) = (y_0 + a/r)e^{rt} - a/r$$

Application 1: Radioactive Decay

$N(t)$: $\sharp$ of radioactive atoms

- Model: $dN/dt \sim -N$

$$\Rightarrow \quad dN/dt = -\lambda N$$

- Solution: $N(t) = N_0 e^{-\lambda t}$
- Half–life:

$$N(t)/N(0) = e^{-\lambda t} = 1/2$$

$$\Rightarrow \quad t = (\ln 2)/\lambda \equiv T_{1/2}$$

- Natural log of ratios:

$$\ln[N_0/N(t)] = \lambda t$$

Ex. 25: After $t = 4\,hrs$, $80\,mg$ of a $100\,mg$ sample of Tritium remain. Determine λ and $T_{1/2}$.

- Use $\lambda = (1/t)\ln[N_0/N(t)]$ to determine λ from measurement
- Use $t = (1/\lambda)\ln[N_0/N(t)]$ to determine time t^* s.t. $N(t^*) = N^*$ for given N^*

Answer:
$\lambda = (1/4)\ln[100/80] = 0.056/hrs$
$T_{1/2} = (\ln 2)/0.056 = 12.43\,hrs$

Ex. 26: $T_{1/2} = 6\,hrs$ for Technitium $99m$. What remains after $9\,hrs$ if $N_0 = 10\,g$?

Answer:
$\lambda = (\ln 2)/6 = 0.116/hr$
$\Rightarrow N(9) = 10e^{-0.116\times 9} = 3.54\,g$

Application 2: Newton's Law of Cooling

$T(t)$: temperature of objet

A: surrounding temperature

- Model: $dT/dt \sim A - T$

$$\Rightarrow \quad dT/dt = k(A - T)$$

- Solution (see Example p.5):

$$T(t) = A + e^{-kt}(T_0 - A)$$

- $(T - A)/(T_0 - A) = e^{-kt} \quad \Rightarrow$

$$kt = \ln[(T_0 - A)/(T(t) - A)]$$

$k = (1/t)\ln[(T_0 - A)/(T(t) - A)]$
$\rightarrow$ determine k

$t = (1/k)\ln[(T_0 - A)/(T(t) - A)]$
$\rightarrow$ determine t

Ex. 33: Dead body found at $t = 0$ (midnight). Temperature 31^0C.
1 hr later ($1\,am$): Temperature 29^0C
Surrounding temperature: $A = 21^0C$
Question: When did death (murder) occur?

Answer: $t = 1\,hr$, $T_0 = 31$, $T(1) = 29$
$\Rightarrow k = (1/1)\ln[(31 - 21)/(29 - 21)]$
$= 0.223/hr$

Determine time at which $T = 37$:
$t = (1/k)\ln[(31 - 21)/(37 - 21)]$
$= -2.11\,hrs = -2\,hrs\,7\,min$

$\Rightarrow$ Death occured at $9:53\,pm$