Chapter 2: First Order ODEs

2.1: ODEs and Solutions

y: dependent variable (or $x, T, P, \ldots$)
t: independent variable (or $x, \omega, s, \ldots$)
$y' = \frac{dy}{dt}$: rate of change

General (implicit) first order ODE:

$$\Phi(t, y, y') = 0 \tag{1}$$

Normal Form:

$$y' = f(t, y) \tag{2}$$

Solution: Function $y(t)$ s.t. (1) or (2) holds identically in t, where $y(t)$ is defined.

Implicit Solution: Relation $\Psi(t, y) = 0$ s.t.

$$\frac{d}{dt}\Psi(t, y) = \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial y} y' = 0.$$

Note: $y'' + 4y = 0$ is 2nd order ODE.

Examples for (1):

$$\begin{aligned} t + 4yy' &= 0 \rightarrow y' = -\frac{t}{4y} \\ y'^2 + y^2 &= 1 \rightarrow y' = \pm\sqrt{1 - y^2} \\ e^{y'} + yy' &= 0 \rightarrow \text{no analytical normal form} \end{aligned}$$

Examples for (2):

$$\begin{aligned} y' &= y - t \\ y' &= y^2 - t \\ y' &= \cos(ty) \\ y' &= y^3 \end{aligned}$$

Check Solutions

Ex.: $y' = y - t$

Claim: $y(t) = t + 1$
is solution on $(-\infty, \infty)$.

Proof: $y' = 1$, $y - t = 1$
(both sides coincide)

Ex.: $y' = -2ty$

Claim: $y(t) = Ce^{-t^2}$
is general solution (arbitrary C).

Proof:
$y' = -2Cte^{-t^2}$, $-2ty = -2Cte^{-t^2}$

Initial Value Problem:

$$\begin{aligned} y' &= f(t, y) \\ y(t_0) &= y_0 \end{aligned}$$

Interval of Existence:

Largest interval containing t_0 on which solution of IVP exists.

Ex.: $y' = y^2$

Given $y(t) = \frac{1}{C-t}$ is a general solution, find the particular solution s.t. $y(0) = 1$

Answer: $y(0) = 1/C = 1 \Rightarrow C = 1$
$\Rightarrow \quad y(t) = \frac{1}{1-t}$, I.o.E.: $(-\infty, 1)$

Worked out Examples from Exercises

Ex. 1: $\Phi(x,y,z) \equiv x^2 z + (1+x)y = 0$

$z = y' \Rightarrow x^2 y' + (1+x)y = 0$

$\Rightarrow$ normal form: $y' = -\frac{1+x}{x^2} y$

Ex. 5: $y' + y/2 = 2\cos t$

Claim:

$y(t) = \frac{4}{5}\cos t + \frac{8}{5}\sin t + Ce^{-t/2}$

is a general solution.

Proof:

$$\begin{aligned} & y' + y/2 \\ = \; & (-\frac{4}{5}\sin t + \frac{8}{5}\cos t - \frac{C}{2}e^{-t/2}) \\ & +(\frac{2}{5}\cos t + \frac{4}{5}\sin t + \frac{C}{2}e^{-t/2}) \\ = \; & 2\cos t \end{aligned}$$

Ex. 9: Implicit ODE: $t - 4yy' = 0$

(a) Claim: $\Psi(t,y) \equiv t^2 - 4y^2 = C^2$

defines implicit solution.

Proof:

$\frac{d}{dt}\Psi(t,y) = \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial y} y' = 2t - 8yy' = 0$

(b) Find explicit solutions.:

$$t^2 - 4y^2 = C^2$$

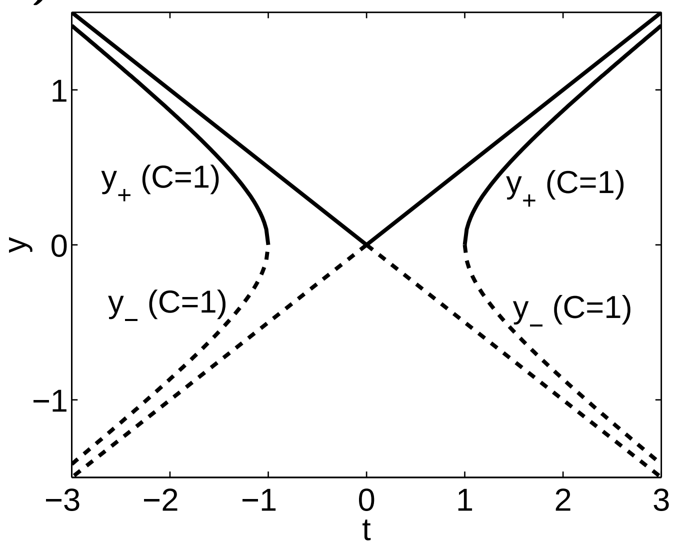
$$\Rightarrow y = \pm\tfrac{1}{2}\sqrt{t^2 - C^2} \equiv y_\pm(t)$$

Verify $y_\pm$ are solutions:

$$\begin{aligned} & t - 4y_\pm y'_\pm \\ = \; & t - 4(\pm\frac{1}{2}\sqrt{t^2 - C^2})(\pm\frac{1}{2}\frac{1}{2\sqrt{t^2 - C^2}}2t) \\ = \; & t - t = 0 \end{aligned}$$

(c) IoE: $t^2 > C^2$
$\Rightarrow (-\infty, -|C|)$ and $(|C|, \infty)$

(d)



Ex. 11: IVP: $y' = y(4-y)$, $y(0) = -1$

Claim: $y(t) = 4/(1-5e^{-4t})$ is solution

Proof: (i) $y(0) = -1$

(ii)

$$\begin{aligned} y' &= [-4/(1-5e^{-4t})^2](-5e^{-4t})(-4) \\ &= -80e^{-4t}/(1-5e^{-4t})^2 \\ y(4-y) &= [4/(1-5e^{-4t})][4-4/(1-5e^{-4t})] \\ &= [4/(1-5e^{-4t})][-20e^{-4t}/(1-5e^{-4t})] \\ &= -80e^{-4t}/(1-5e^{-4t})^2 \end{aligned}$$

IoE: $1 = 5e^{-4t} \Rightarrow t^* = (\ln 5)/4$
$\Rightarrow (-\infty, (\ln 5)/4)$

Ex. 13: IVP: $ty' + y = t^2$, $y(1) = 2$

Given general solution $y(t) = t^2/3 + C/t$, find solution to IVP.

Answer: $y(1) = 1/3 + C = 2 \Rightarrow C = 5/3$
$\Rightarrow y(t) = t^2/3 + 5/3t$; IoE: $(0, \infty)$

Geometric Interpretation

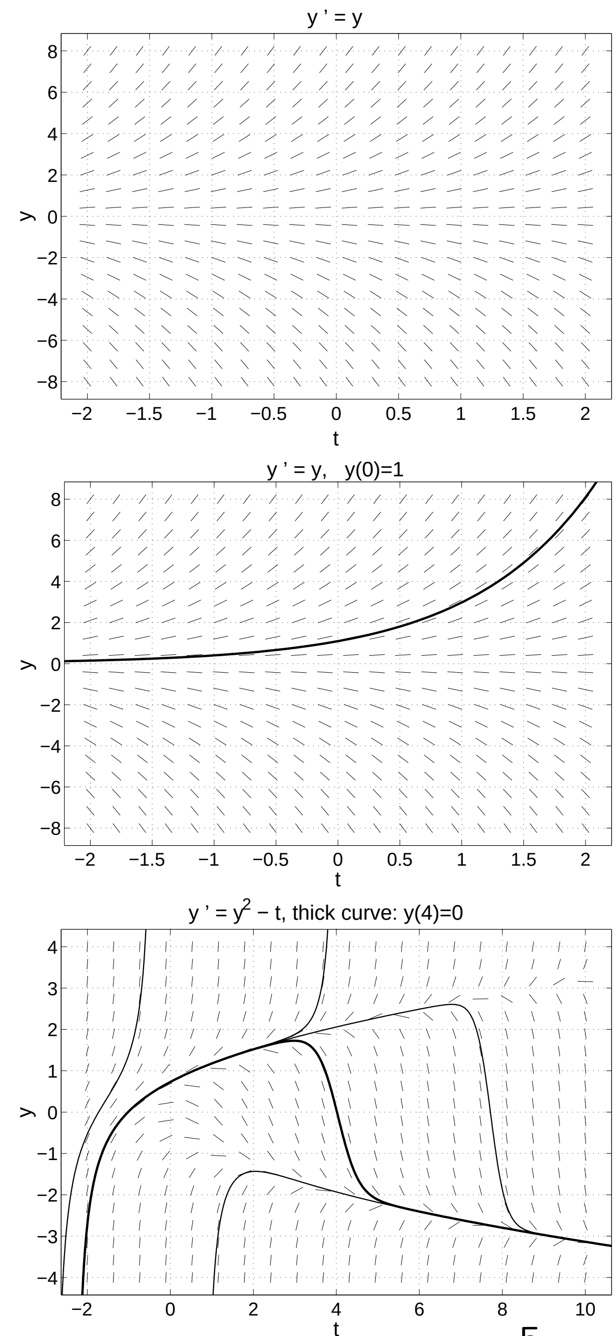
$$\text{ODE:} \quad y' = f(t, y)$$

Graph of solution $y(t)$
= Solution Curve

$$y'(t) = f(t, y) = \text{slope at } (t, y)$$

Direction Field:

- Choose grid points (t_i, y_j) in a rectangle
- At each grid point draw a line segment with slope $f(t_i, y_j)$ (Matlab: `dfield6`)
- $\Rightarrow$ Information about qualitative form of solution curves



Autonomous ODEs

$$y' = f(y)$$

$\Rightarrow$ Same slopes on horizontal lines in (t, y)–plane

Example: $y' = 1 - y^2$

Equilibrium Points:

$f(y) = 0 \rightarrow$ solutions y^*

$y(t) = y^* =$ const
is constant solution of ODE

Example: $f(y) = 1 - y^2 = 0$
$\Rightarrow y_1^* = 1,\ y_2^* = -1.$