

Linear Systems of Algebraic Equations

Section 1: Row operations and reduced row echelon form

Section 2: Determined Systems ($n \times n$ matrices)

2.1: Review of basic theory: singular and nonsingular matrices, linear dependence/independence of vectors, determinants, nullspace and basis

2.2: Matlab computations for a singular (Ex. 1) and a nonsingular (Ex. 2) matrix

1 Row operations and reduced row echelon form

Consider the following system of equations,

$$\begin{array}{rccccrcr} 3x_1 & - & 4x_2 & + & 5x_3 & - & x_4 & = & 5 \\ & & x_2 & - & x_3 & + & 6x_4 & = & 0 \\ 5x_1 & & & - & x_3 & + & 4x_4 & = & 4 \end{array} \quad (1)$$

for x_1, x_2, x_3, x_4 . Because the left hand sides are sums of multiples of the unknowns, (1) is called a linear system of equations. To find *all* solutions of these equations, we successively eliminate variables as follows:

(1) Divide the first equation by 3 and leave the other equations unchanged:

$$\begin{array}{rccccrcr} x_1 & - & \frac{4}{3}x_2 & + & \frac{5}{3}x_3 & - & \frac{1}{3}x_4 & = & \frac{5}{3} \\ & & x_2 & - & x_3 & + & 6x_4 & = & 0 \\ 5x_1 & & & - & x_3 & + & 4x_4 & = & 4 \end{array}$$

(2) Add -5 times the first equation to the third equation:

$$\begin{array}{rccccrcr} x_1 & - & \frac{4}{3}x_2 & + & \frac{5}{3}x_3 & - & \frac{1}{3}x_4 & = & \frac{5}{3} \\ & & x_2 & - & x_3 & + & 6x_4 & = & 0 \\ \frac{20}{3}x_2 & - & \frac{28}{3}x_3 & + & \frac{17}{3}x_4 & = & -\frac{13}{3} \end{array}$$

(3) Add $-20/3$ times the second equation to the third equation and add $4/3$ times the second equation to the first equation:

$$\begin{array}{rccccrcr} x_1 & & + & \frac{1}{3}x_3 & + & \frac{23}{3}x_4 & = & \frac{5}{3} \\ x_2 & - & x_3 & + & 6x_4 & = & 0 \\ & & - & \frac{8}{3}x_3 & - & \frac{103}{3}x_4 & = & -\frac{13}{3} \end{array}$$

(4) Multiply third equation by $-3/8$:

$$\begin{array}{rccccrcr} x_1 & & + & \frac{1}{3}x_3 & + & \frac{23}{3}x_4 & = & \frac{5}{3} \\ x_2 & - & x_3 & + & 6x_4 & = & 0 \\ & & x_3 & + & \frac{103}{8}x_4 & = & \frac{13}{8} \end{array}$$

(5) Add second equation to third equation and add $-1/3$ times third equation to first equation:

$$\begin{aligned} x_1 & & + \frac{27}{8} x_4 & = \frac{9}{8} \\ x_2 & & + \frac{151}{8} x_4 & = \frac{13}{8} \\ x_3 & + \frac{103}{8} x_4 & = \frac{13}{8} \end{aligned} \tag{2}$$

In this form we can easily find all solutions. Obviously x_4 can be given any number, but if x_4 is fixed, x_1, x_2, x_3 are also fixed. We set $x_4 = t$ and solve for x_1, x_2, x_3 to obtain the following one parameter family of solutions,

$$\begin{aligned} x_1 & = \frac{9}{8} - \frac{27}{8} t \\ x_2 & = \frac{13}{8} - \frac{151}{8} t \\ x_3 & = \frac{13}{8} - \frac{103}{8} t \\ x_4 & = t \end{aligned} \tag{3}$$

where t is an arbitrary number.

Let's redo these computations using matrices and vectors. In matrix notation (1) can be written as

$$\begin{bmatrix} 3 & -4 & 5 & -1 \\ 0 & 1 & -1 & 6 \\ 5 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} \tag{4}$$

which has the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & -4 & 5 & -1 \\ 0 & 1 & -1 & 6 \\ 5 & 0 & -1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}.$$

The matrix A is the coefficient matrix, \mathbf{b} is the target vector and \mathbf{x} is the vector of unknowns. We augment A by appending \mathbf{b} as 5th column to A and denote the resulting augmented matrix by M ,

$$M = \begin{bmatrix} 3 & -4 & 5 & -1 & 5 \\ 0 & 1 & -1 & 6 & 0 \\ 5 & 0 & -1 & 4 & 4 \end{bmatrix}.$$

We can repeat the above manipulations by operating on the rows of M . For example, dividing the first three equations by 3 is equivalent to dividing all entries in the first row by 3. Similarly, adding -5 times the first equation to the third equation is equivalent to adding -5 times the first row to the third row etc.

We will use Matlab to do these computations. Since all entries of M are rational (in fact, integers), we switch to the rational format by typing `format rat` at the Matlab prompt. From this point on Matlab will display all numbers as ratios of integers. But be aware that also in rational format Matlab is dealing with approximations all the time. For example, when executing

```
>> format rat
>> pi
```

```
ans =
```

```
355/113
```

the value of π (which is irrational) is *approximated* by a rational number instead of a decimal number.

First type in A , b and define the matrix M :

```
>> A=[3 -4 5 -1;0 1 -1 6;5 0 -1 4];b=[5 0 4]';M=[A,b]
```

```
M =
```

```
3         -4         5         -1         5
0          1         -1         6         0
5          0         -1         4         4
```

To reduce the effect of roundoff errors as much as possible, we refer to the matrix elements by names instead of the displayed numbers.

(1) $M(1,1)$ is nonzero. Normalize $M(1,1)$ to 1 by dividing the first row by this entry:

```
>> M(1,:)=M(1,:)/M(1,1)
```

```
M =
```

```
1         -4/3         5/3         -1/3         5/3
0          1          -1          6          0
5          0          -1          4          4
```

(2) Subtract $M(3,1)$ times the first row from the third row so that $M(3,1)$ transforms to zero:

```
M(3,:)=M(3,:)-M(3,1)*M(1,:)
```

```
M =
```

```
1         -4/3         5/3         -1/3         5/3
0          1          -1          6          0
0         20/3        -28/3         17/3        -13/3
```

(3) Transform $M(1,2)$ to zero by subtracting $M(1,2)$ times the 2nd row from the first row. Similarly transform $M(3,2)$ to zero by subtracting $M(3,2)$ times the second row from the third row:

```
M(1,:)=M(1,:)-M(1,2)*M(2,:);M(3,:)=M(3,:)-M(3,2)*M(2,:)
```

M =

$$\begin{array}{ccccc} 1 & 0 & 1/3 & 23/3 & 5/3 \\ 0 & 1 & -1 & 6 & 0 \\ 0 & 0 & -8/3 & -103/3 & -13/3 \end{array}$$

(4) Normalize $M(3,3)$ to 1 by dividing the third row by this entry:

$$M(3,:) = M(3,)/M(3,3)$$

M =

$$\begin{array}{ccccc} 1 & 0 & 1/3 & 23/3 & 5/3 \\ 0 & 1 & -1 & 6 & 0 \\ 0 & 0 & 1 & 103/8 & 13/8 \end{array}$$

(5) Finally transform $M(1,3)$ and $M(2,3)$ to zero by adding appropriate multiples of the third row to the first and second rows:

$$M(1,:) = M(1,)-M(1,3)*M(3,); M(2,:) = M(2,)-M(2,3)*M(3,)$$

M =

$$\begin{array}{ccccc} 1 & 0 & 0 & 27/8 & 9/8 \\ 0 & 1 & 0 & 151/8 & 13/8 \\ 0 & 0 & 1 & 103/8 & 13/8 \end{array}$$

The new matrix M , interpreted as augmented matrix of a linear system of equations (last column = target vector, first three columns = coefficient matrix), is equivalent to equations (2) from which we easily could read off the general, 1-parameter family of solutions given by (3). This form of M is called the reduced row echelon form of the original matrix. In general a matrix is said to be in reduced row echelon form if it has the form

$$\begin{bmatrix} 1 & 0 & \# & \# & 0 & 0 & \# & \# & 0 & \# \\ 0 & 1 & \# & \# & 0 & 0 & \# & \# & 0 & \# \\ 0 & 0 & 0 & 0 & 1 & 0 & \# & \# & 0 & \# \\ 0 & 0 & 0 & 0 & 0 & 1 & \# & \# & 0 & \# \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \# \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5)$$

where the #'s stand for arbitrary numbers. A more formal definition is the following:

Define the pivot of a row to be the first nonzero entry in that row (to the left of the pivot all entries of the row are zero). A matrix is in reduced row echelon form if it has the following properties:

- There may be rows consisting entirely of zeros (i.e. rows without pivots), but these rows occur at the bottom of the matrix.
- Each pivot is a 1.

- Each pivot is in a column *strictly to the right* of the pivots occurring in the rows above it.

The strategy for solving linear systems of equations amounts to transforming the augmented matrix of the system to reduced row echelon form using elementary row operations. This method is called Gaussian elimination. The admitted elementary operations are:

- Multiply a row by a nonzero number.
- Add a multiple of a row to another row.
- Flip two rows.

By applying an appropriate sequence of such operations in succession one can always transform a matrix to reduced row echelon form. If a matrix is in that form, a column which contains a pivot is called a pivot column and a column which contains no pivot is called a free column. For the associated system of equations the variables corresponding to the free columns are called free variables. To these variables one can assign arbitrary values. The variables corresponding to the pivot columns are called pivot variables. They depend on the free variables, i.e. if values for the free variables have been chosen the values of the pivot variables are determined. In our example the fourth column - corresponding to the variable x_4 - is a free column while the first, second and third columns are pivot columns. The fifth column is the transformed target vector.

In the example the transformation to reduced row echelon form was carried out step by step to illustrate the elementary row operations. To go through all these steps is not necessary. Matlab provides the command `rref` to do the whole process at once:

```
>> rref(M)
```

```
ans =
```

```

     1         0         0     27/8         9/8
     0         1         0    151/8        13/8
     0         0         1    103/8        13/8
```

Let's do another example:

$$\begin{array}{rcl}
 & 2x_2 + 2x_3 + 3x_4 & = -4 \\
 - & 2x_1 + 4x_2 + 2x_3 - x_4 & = -6 \\
 & 3x_1 - 4x_2 - x_3 + 24x_4 & = 8
 \end{array} \tag{6}$$

The augmented matrix is:

```
>> M=[0 2 2 3 -4;-2 4 2 -1 -6;3 -4 -1 2 8]
```

```
M =
```

| | | | | |
|----|----|----|----|----|
| 0 | 2 | 2 | 3 | -4 |
| -2 | 4 | 2 | -1 | -6 |
| 3 | -4 | -1 | 2 | 8 |

Place the augmented matrix in reduced row echelon form:

```
>> rref(M)
```

```
ans =
```

| | | | | |
|---|---|---|---|------|
| 1 | 0 | 1 | 0 | 16/5 |
| 0 | 1 | 1 | 0 | -1/5 |
| 0 | 0 | 0 | 1 | -6/5 |

We see that x_1, x_2, x_4 are pivot variables and x_3 is a free variable. Setting again $x_3 = t$, all solutions of (6) can be parametrically represented by

$$x_1 = 16/5 - t, \quad x_2 = -1/5 - t, \quad x_3 = t, \quad x_4 = -6/5.$$

It is sometimes useful to write the solution also in vector notation:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 - t \\ -1/5 - t \\ t \\ -6/5 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -1/5 \\ 0 \\ -6/5 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

2 Determined systems of equations

The examples of the preceding section are examples of underdetermined systems because there are fewer equations than unknowns. Systems with more equations than unknowns are called overdetermined systems and systems with the same number of equations and unknowns are called determined systems. The reduced row echelon form can be applied to all three types of systems. In ODEs we will mainly be concerned with determined systems. In this case the coefficient matrix is a square ($n \times n$) matrix. For determined (as well as for underdetermined and overdetermined) systems there are essentially three things which can happen: the system has no solution, there is an infinite number of solutions or the system has a unique solution.

2.1 Basic theory

Singular and nonsingular matrices

Let A be a square matrix. The important fact about determined systems is that, if $A\mathbf{x} = \mathbf{b}_0$ has a unique solution for a single target vector \mathbf{b}_0 , then $A\mathbf{x} = \mathbf{b}$ has a unique solution for *every* target vector \mathbf{b} . A particular target vector is the zero vector, $\mathbf{b}_0 = \mathbf{0}$, whose entries are all 0. Since clearly $A\mathbf{0} = \mathbf{0}$ holds, the system $A\mathbf{x} = \mathbf{0}$ (called the homogeneous system of equations associated with A) always has the (trivial) solution $\mathbf{x} = \mathbf{0}$. Consequently, if $\mathbf{x} = \mathbf{0}$ is the *only solution* that $A\mathbf{x} = \mathbf{0}$ admits, $A\mathbf{x} = \mathbf{b}$ has a unique solution for any target vector \mathbf{b} . In this case, because of linearity, the solution can be represented through a matrix as $\mathbf{x} = B\mathbf{b}$. The matrix B is called the inverse of A and is denoted by $B = A^{-1}$. The inverse matrix satisfies $AA^{-1} = A^{-1}A = I$, where I is the identity matrix which has ones in the diagonal while all other entries are zero.

In contrast, if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, different from the zero vector, there are vectors \mathbf{b} for which $A\mathbf{x} = \mathbf{b}$ admits no solution, and for those vectors \mathbf{b} for which a solution exists there are infinitely many solutions. A square matrix A for which $A\mathbf{x} = \mathbf{0}$ admits nontrivial (and hence infinitely many) solutions is called a *singular matrix* and a square matrix for which $A\mathbf{x} = \mathbf{0}$ admits only the trivial solution (A^{-1} exists) is called a *nonsingular matrix*.

Linear dependence and linear independence of vectors.

A set of m (column or row) vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ is called linearly dependent if there are numbers c_1, c_2, \dots, c_m , not all zero, such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \dots + c_m\mathbf{a}_m = \mathbf{0}. \quad (7)$$

If no such numbers exist, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ are called linearly independent. Assume now that these vectors are column vectors. Recall that multiplication of a matrix to a column vector yields the linear combination of the columns of the matrix associated with the entries of the vector to which the matrix is multiplied. Hence (7) is the same as

$$A\mathbf{c} = \mathbf{0}, \quad (8)$$

in matrix–vector notation, where $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m]$ is the $m \times n$ –matrix whose columns are formed by $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$, and

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}.$$

Thus the vectors are linearly dependent if the homogeneous system of equations (8) associated with A admits a nontrivial solution. Therefore the question of linear dependence or independence of the vectors can be answered by inspecting the reduced row echelon form of A (for homogeneous systems of equations it is not necessary to augment the matrix by the target vector $\mathbf{0}$ because all row operations will leave the zero vector unchanged). If the reduced row echelon form has free columns, we can give the corresponding free variables arbitrary nonzero values and solve for the pivot variables to obtain nontrivial solutions. Hence the vectors are linearly independent if (and only if) the reduced row echelon form of the associated matrix A has a free column. When $m > n$ the maximal number of pivots is n (since there are n rows), but there are m , i.e. more than n columns, thus there is at least one free column. In this case the vectors will necessarily be linearly dependent. For $m \leq n$ there may or may not be a free column which is equivalent to $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ being linearly dependent or linearly independent.

If the vectors are linearly independent, (8) admits only the trivial solution. In particular for $m = n$ this means that the matrix formed by these column vectors is nonsingular. This is equivalent to saying that the reduced row echelon form is the identity matrix because then all n columns are pivot columns. Conversely, given a square matrix A , A is nonsingular (singular) if and only if its columns are linearly independent (linearly dependent) vectors. One can show that the same holds with the columns replaced by the rows.

Determinants.

To a square matrix A one can associate a single number whose value tells whether A is singular or nonsingular: the determinant $\det(A) = |A|$. If $|A| \neq 0$, A is nonsingular (A^{-1} exists) and if $|A| = 0$, A is singular. For 2×2 –matrices the determinant can be easily calculated:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For 3×3 –matrices the formula for the determinant still can be written down,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + cdh + bfg - ceg,$$

but for larger matrices the formulae become increasingly lengthy. For hand calculations you may find the row or column expansion formula (text: section 5.1, equation (20), p. 289) useful. There is a special case in which the computation of the determinant becomes extremely easy: the case of lower (or upper) triangular matrices. A matrix is called lower

(upper) triangular if the entries above (below) the diagonal are all zero, e.g.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \quad \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}.$$

For such matrices the determinant is simply the product of the diagonal entries (adf for the example).

Let's summarize these facts in a theorem:

Theorem: Given a square matrix A , the following statements are equivalent:

- $|A| \neq 0$.
- The columns and rows of A are linearly independent.
- The reduced row echelon form of A is the identity matrix.
- The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
- A is nonsingular, i.e. A^{-1} exists.

The contrary equivalent statements are:

- $|A| = 0$.
- The columns and rows of A are linearly dependent.
- The reduced row echelon form of A has a free column.
- $A\mathbf{x} = \mathbf{0}$ admits a nontrivial solution (different from $\mathbf{0}$).
- A is singular, i.e. A^{-1} does not exist.

Nullspace of a matrix.

The nullspace of a matrix A is defined as the set of all vectors \mathbf{x} such that $A\mathbf{x} = \mathbf{0}$. Because $\mathbf{x} = \mathbf{0}$ is always a solution, the zero vector is always an element of the nullspace. Let's assume A is a square matrix. Then the theorem above tells that if A is nonsingular, $\mathbf{x} = \mathbf{0}$ is the only element of the nullspace. On the other hand, if the square matrix A is singular, there are lot's of vectors in the nullspace. These vectors can be found from the reduced row echelon form. If the reduced row echelon form has p free variables, we can assign to these variables arbitrary parameters t_1, \dots, t_p . Solving then for the pivot variables and writing the solution in vector form yields a representation of the general solution to $A\mathbf{x} = \mathbf{0}$ in the form

$$\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \dots + t_p\mathbf{x}_p, \tag{9}$$

with certain vectors $\mathbf{x}_1, \dots, \mathbf{x}_p$. Hence any vector in the nullspace of A can be written as a linear combination of p vectors and one can show that these vectors are linearly independent.

Definition: A basis of the nullspace of a singular square matrix A is a set of linearly independent vectors $\mathbf{x}_1, \dots, \mathbf{x}_p$ such every vector in the nullspace can be represented in the form (9). The number p of these vectors is called the dimension of the nullspace.

2.2 Examples

Example 1

$$A = \begin{bmatrix} 12 & -8 & -4 & 30 \\ 3 & -2 & -1 & 6 \\ 18 & -12 & -6 & 42 \\ 6 & -4 & -2 & 15 \end{bmatrix}.$$

Define A in Matlab:

```
>> A=[12 -8 -4 30;3 -2 -1 6;18 -12 -6 42;6 -4 -2 15];
```

We first want to check if the matrix is singular or nonsingular, so let's compute the determinant:

```
>> det(A)
```

```
ans =
```

```
0
```

The determinant is zero, hence the matrix is singular. We first determine the nullspace of A and then investigate nonhomogeneous systems of equations associated with this matrix.

(1) Nullspace

(a) *Using reduced row echelon form.* Let's find the solutions to $A\mathbf{x} = \mathbf{0}$ by computing the reduced row echelon form of A :

```
>> rref(A)
```

```
ans =
```

```
1      -2/3      -1/3      0
0         0         0         1
0         0         0         0
0         0         0         0
```

The associated homogeneous system of equations for this form is:

$$\begin{aligned} x_1 - (2/3)x_2 - (1/3)x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

The reduced row echelon form has the first and fourth column as pivot columns and the second and third columns as free columns. Accordingly we consider x_2, x_3 as free variables and x_1, x_4 as pivot (dependent) variables. Thus we set $x_2 = s, x_3 = t$ and solve for x_1 and x_4 :

$$x_1 = (2/3)s + (1/3)t, \quad x_2 = s, \quad x_3 = t, \quad x_4 = 0.$$

Write this in vector form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (2/3)s + (1/3)t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} (2/3) \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} (1/3) \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The last expression in this formula represents a vector in the nullspace of A in the form $\mathbf{x} = s\mathbf{x}_1 + t\mathbf{x}_2$, where

$$\mathbf{x}_1 = \begin{bmatrix} (2/3) \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} (1/3) \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

and s, t are arbitrary parameters. Hence the vectors $\mathbf{x}_1, \mathbf{x}_2$ form a basis of the nullspace, i.e. the dimension of the nullspace is 2. Since the dimension of the nullspace coincides with the number of free parameters, it can be directly read off the reduced row echelon form (just count the number of free columns).

(b) *Direct Computation of nullspace.* Matlab allows you to directly compute a basis for the nullspace of A , i.e. you don't need to first compute the reduced row echelon form and then derive from this the basis vectors. The command *null* causes Matlab to do the whole process at once. Let's try it for our matrix A :

```
>> nullA=null(A)
```

```
nullA =
```

```

531/889      -194/10023
401/571      -434/923
545/1407     6277/7114
      *           *
```

This output is not what we expect regarding our previous result. The reason is that *null*, without option, attempts an approximation of basis vectors with special properties and displays them in the prescribed format. The asteriks stand for lengthy numbers which are not displayed, but we can access them:

```
>> nullA(:,1)
```

```
ans =
```

```

531/889
401/571
545/1407
1/1493919469786894
```

Obviously the first asterik represents a very small number which should be zero. The same holds for the second asterik. You can see this by switching to the long format:

```
>> format long>null(A)
```

```
ans =
```

```
0.59730078067420 -0.01935547852286
0.70227662297112 -0.47020554985453
0.38734909608036 0.88234466414047
0.00000000000000 0.00000000000000
```

To obtain a nice output, suitable for computations with rational numbers, Matlab offers the option 'r'. With this option the basis for the nullspace is computed in a manner similar to the procedure we have used:

```
>> format rat>null(A,'r')
```

```
ans =
```

```
2/3      1/3
1         0
0         1
0         0
```

These vectors are exactly the basis vectors which we have found. You can use the option 'r' also in other formats, e.g.:

```
>> format long>null(A,'r')
```

```
ans =
```

```
0.66666666666667 0.33333333333333
1.00000000000000 0
0 1.00000000000000
0 0
```

(2) Nonhomogeneous systems

Next let us investigate linear systems of equations associated with A for nonzero target vectors. We consider two vectors:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 30 \\ 6 \\ 42 \\ 15 \end{bmatrix}.$$

Example with no solutions: We first investigate the equation $A\mathbf{x} = \mathbf{b}_1$. Define \mathbf{b}_1 in Matlab and set up the augmented matrix:

```
>> b=[1;2;3;4];M=[A b]
```

```
M =
```

| | | | | |
|----|-----|----|----|---|
| 12 | -8 | -4 | 30 | 1 |
| 3 | -2 | -1 | 6 | 2 |
| 18 | -12 | -6 | 42 | 3 |
| 6 | -4 | -2 | 15 | 4 |

Place the matrix in reduced row echelon form:

```
>> rref(M)
```

```
ans =
```

| | | | | |
|---|------|------|---|---|
| 1 | -2/3 | -1/3 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 |

The third equation of the linear system of equations associated with this form reads

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

which cannot be satisfied by any vector $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$. Thus the system $A\mathbf{x} = \mathbf{b}_1$ does not admit solutions. You can clearly see why there is no solution from the reduced row echelon form: the last column is a pivot column.

Theorem: The linear system of equations $A\mathbf{x} = \mathbf{b}$ has a solution (no solutions) if and only if the last column of the reduced row echelon form of the augmented matrix $[A, \mathbf{b}]$ is a free column (pivot column).

Example with infinitely many solutions: Now do the same for the vector \mathbf{b}_2 :

```
>> b=[30;6;42;15];M=[A b]
```

```
M =
```

| | | | | |
|----|-----|----|----|----|
| 12 | -8 | -4 | 30 | 30 |
| 3 | -2 | -1 | 6 | 6 |
| 18 | -12 | -6 | 42 | 42 |
| 6 | -4 | -2 | 15 | 15 |

```
>> rref(M)
```

```
ans =
```

| | | | | |
|---|------|------|---|---|
| 1 | -2/3 | -1/3 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 |

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

In this case we are lucky: the last column is not a pivot column, hence the system admits solutions. The second and third columns are free columns, thus we set again $x_2 = s$, $x_3 = t$. Then the first and second rows of the reduced row echelon form are equivalent to the equations

$$x_1 - (2/3)s - (1/3)t = 0, \quad x_4 = 1,$$

for the pivot variables x_1, x_2 . Thus the general solution is:

$$x_1 = (2/3)s + (1/3)t, \quad x_2 = s, \quad x_3 = t, \quad x_4 = 1,$$

Write this in vector form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (2/3)s + (1/3)t \\ s \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} (2/3) \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} (1/3) \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Notice that the vectors multiplying s and t are the basis vectors of the nullspace of A found before. Setting $s = t = 0$ in the general solution leaves us with the solution vector $\mathbf{x} = \mathbf{x}_p$, where

$$\mathbf{x}_p = [0, 0, 0, 1]^T.$$

Hence the general solution of our (nonhomogenous) system $A\mathbf{x} = \mathbf{b}_2$ has the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h = s\mathbf{x}_1 + t\mathbf{x}_2$ is the general solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, and \mathbf{x}_p is a particular solution of the nonhomogenous system. This decomposition is a general property of solutions of linear systems of algebraic equations.

Example 2

$$A = \begin{bmatrix} 3 & -4 & -8 \\ 0 & 2 & 2 \\ 1 & -2 & -3 \end{bmatrix}.$$

We first check whether A is singular or nonsingular:

```
>> A=[3 -4 -8;0 2 2;1 -2 -3];det(A)
```

```
ans =
```

```
2
```

The determinant is nonzero, hence A is nonsingular. Given a target vector \mathbf{b} , the system $A\mathbf{x} = \mathbf{b}$ should have a unique solution.

Solution of a nonhomogeneous system and inverse matrix

Let's check uniqueness of the solution of $A\mathbf{x} = \mathbf{b}$ for

$$\mathbf{b} = \begin{bmatrix} 27 \\ -6 \\ 10 \end{bmatrix}.$$

(a) *Using reduced row echelon form:* As usual define first the augmented matrix,

```
>> b=[27 -6 10]';M=[A b]
```

```
M =
```

```
     3     -4     -8     27
     0      2      2     -6
     1     -2     -3     10
```

and transform it to reduced row echelon form:

```
rref(M)
```

```
ans =
```

```
     1      0      0      1
     0      1      0      0
     0      0      1     -3
```

The system of equations associated with the reduced row echelon form yields a unique solution as expected,

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = -3.$$

Note that the solution vector appears directly in the last column of the reduced row echelon form. This is clear because the matrix of the first three columns is the reduced row echelon form of A which in case of nonsingular matrices is just the identity matrix. In matrix-vector form the associated system of equations reads $I\mathbf{x} = \tilde{\mathbf{b}}$ where $\tilde{\mathbf{b}}$ is the last column (the transformed target vector), hence $\mathbf{x} = \tilde{\mathbf{b}}$. Thus, if it is guaranteed that A is nonsingular, the commands

```
>> rref([A b]);x=ans(:,4);x'
```

```
ans =
```

```
     1      0     -3
```

provide us directly with the solution vector (displayed as row to save space).

(b) Direct Computation of solution: The reduced row echelon form is a powerful concept which gives insight into the solution structure of any linear system of equations. In case of a nonsingular matrix, Matlab offers two ways to find the solution directly through simple commands. First there is the “left matrix division operation” executed with a backslash:

```
>> A\b
```

```
ans =
```

```
     1
1/1125899906842624
    -3
```

The very large denominator in the second term indicates that Matlab is approximating 0 (look at this answer in long format). Thus, this solution agrees with the one found before. The second approach is to compute the inverse matrix A^{-1} ,

```
>> invA=inv(A)
```

```
invA =
```

```
    -1         2         4
     1        -1/2       -3
    -1         1         3
```

and to multiply this matrix to the target vector,

```
>> invA*b
```

```
ans =
```

```
     1
1/281474976710656
    -3
```

which yields the same answer as the left matrix division. Both commands give results which make sense only if the matrix is nonsingular, even if a solution exists. For example, for the matrix of Example 1 and the target vector \mathbf{b}_2 we know that there are solutions, however, the Matlab response on left division is not useful:

```
>> [12 -8 -4 30;3 -2 -1 6;18 -12 -6 42;6 -4 -2 15]\[30;6;42;15]
```

```
Warning: Matrix is singular to working precision.
```

```
ans =
```

```
    1/0
    1/0
    1/0
    1/0
```

There are other commands which can be applied in such cases, but these are based on more advanced linear algebra and will not be discussed.

Relation between row reduced echelon form and inverse matrix

Let's elaborate a bit more on the inverse matrix. Denote by \mathbf{x}_1 , \mathbf{x}_2 , \mathbf{x}_3 the first, second and third columns of A^{-1} . Since $AA^{-1} = I$, we have $A\mathbf{x}_1 = \mathbf{e}_1$, $A\mathbf{x}_2 = \mathbf{e}_2$ and $A\mathbf{x}_3 = \mathbf{e}_3$, where

$$\mathbf{e}_1 = [1, 0, 0]^T, \quad \mathbf{e}_2 = [0, 1, 0]^T, \quad \mathbf{e}_3 = [0, 0, 1]^T,$$

are the three coordinate vectors (columns of the identity matrix). Hence the columns of A^{-1} are nothing than solutions of the systems $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = \mathbf{e}_1$, $\mathbf{b} = \mathbf{e}_2$ and $\mathbf{b} = \mathbf{e}_3$. For example,


```
>> rref([A,[0,1,0]'])
```

```
ans =
```

```
    1         0         0         2
    0         1         0        -1/2
    0         0         1         1
```

yields as last column the second column of A^{-1} . If we augment A by all three coordinate vectors we obtain a 3×6 -matrix whose reduced row echelon form contains A^{-1} in columns 4 through 6. This augmented matrix is just $[A, I]$. Recall that the 3×3 -identity matrix is built in in Matlab as `eye(3)`:

```
>> M=[A,eye(3)]
```

```
M =
```

```
    3         -4        -8         1         0         0
    0         2         2         0         1         0
    1        -2        -3         0         0         1
```

```
rref(M)
```

```
ans =
```

```
    1         0         0        -1         2         4
    0         1         0         1        -1/2        -3
    0         0         1        -1         1         3
```

As it should be, the inverse matrix appears in the last three columns. Thus the two commands,

```
>> rref([A,eye(3)]); invA=ans(:,4:6)
```

```
invA =
```

```
   -1         2         4
    1        -1/2        -3
   -1         1         3
```

provide an alternative approach to compute the inverse matrix.

This procedure, i.e. the transformation of $[A, I]$ to reduced row echelon form, is also the recommended method for computing the inverse of a matrix of size greater than 2 by hand. For 2×2 -matrices the recommended method is to apply the formula given in the text (Section 5.1, equation (21), p.290).