9.9: Inhomogeneous Systems

Generalize Ch.2: $x' = a(t)x + f(t) \to x_p(t) = x_h(t)\int [f(t)/x_h(t)]dt$, $x_h' = a(t)x_h$

Inhomogeneous system:
$(A(t) : n \times n)$

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t) \qquad (1)$$

Consider also

$$\mathbf{x}' = A(t)\mathbf{x} \qquad (2)$$

Let $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ be F.S.S. of (2) $\Rightarrow$ F.M.:

$$X(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)]$$

Particular solution of (1):

$$\mathbf{x}_p(t) = X(t)\int X(t)^{-1}\mathbf{f}(t)\,dt \qquad (3)$$

or (using definite integral)

$$\mathbf{x}_p(t) = X(t)\int_{t_0}^{t} X(s)^{-1}\mathbf{f}(s)\,ds \qquad (4)$$

$[(4) \Rightarrow \mathbf{x}_p(t_0) = \mathbf{0}]$

General solution of (1):

$$\mathbf{x}(t) = X(t)\mathbf{c} + \mathbf{x}_p(t)$$

Ex. 1: Find general solution of (1) if

$$A = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

Eigenvalue/eigenvector analysis $\Rightarrow$

$$\mathbf{x}_1(t) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{2t}, \quad \mathbf{x}_2(t) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{t}$$

$$\Rightarrow X(t) = \begin{bmatrix} 2e^{2t} & 3e^t \\ -e^{2t} & -2e^t \end{bmatrix} \Rightarrow W(t) = -e^{3t}$$

$$X(t)^{-1} = \frac{1}{-e^{3t}}\begin{bmatrix} -2e^t & -3e^t \\ e^{2t} & 2e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{-2t} & 3e^{-2t} \\ -e^{-t} & -2e^{-t} \end{bmatrix}$$

$$\int X(t)^{-1}\mathbf{f}(t)\,dt = \int \begin{bmatrix} 2e^{-2t} & 3e^{-2t} \\ -e^{-t} & -2e^{-t} \end{bmatrix}\begin{bmatrix} e^t \\ e^t \end{bmatrix} dt$$

$$= \int \begin{bmatrix} 5e^{-t} \\ -3 \end{bmatrix} dt = \begin{bmatrix} -5e^{-t} \\ -3t \end{bmatrix}$$

$$\mathbf{x}_p(t) = \begin{bmatrix} 2e^{2t} & 3e^t \\ -e^{2t} & -2e^t \end{bmatrix}\begin{bmatrix} -5e^{-t} \\ -3t \end{bmatrix} = \begin{bmatrix} -10e^t - 9te^t \\ 5e^t + 6te^t \end{bmatrix}$$

General solution: $\mathbf{x}(t) = X(t)\mathbf{c} + \mathbf{x}_p(t)$

Ex. 5: Find general solution of (1) for $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$

Compute F.S.S.: $\mathbf{x}_1(t) = e^{2t}\begin{bmatrix} \sin t - \cos t \\ \cos t \end{bmatrix}$, $\mathbf{x}_2(t) = e^{2t}\begin{bmatrix} -\cos t - \sin t \\ \sin t \end{bmatrix}$

$$\Rightarrow \; X(t) = e^{2t}\begin{bmatrix} \sin t - \cos t & -\cos t - \sin t \\ \cos t & \sin t \end{bmatrix}$$

$$\Rightarrow \; W(t) = e^{4t}(\sin^2 t - \cos t \sin t + \cos^2 t + \sin t \cos t) = e^{4t}$$

$$\Rightarrow \; X(t)^{-1} = \frac{1}{e^{4t}} e^{2t}\begin{bmatrix} \sin t & \cos t + \sin t \\ -\cos t & \sin t - \cos t \end{bmatrix} = e^{-2t}\begin{bmatrix} \sin t & \cos t + \sin t \\ -\cos t & \sin t - \cos t \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \; \int X(t)^{-1}\mathbf{f}(t)\,dt &= \int e^{-2t}\begin{bmatrix} \sin t & \cos t + \sin t \\ -\cos t & \sin t - \cos t \end{bmatrix}\begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} dt = \int \begin{bmatrix} \cos t + \sin t \\ \sin t - \cos t \end{bmatrix} dt \\ &= \begin{bmatrix} \sin t - \cos t \\ -\cos t - \sin t \end{bmatrix} \end{aligned}$$

$$\Rightarrow \; \mathbf{x}_p(t) = e^{2t}\begin{bmatrix} \sin t - \cos t & -\cos t - \sin t \\ \cos t & \sin t \end{bmatrix}\begin{bmatrix} \sin t - \cos t \\ -\cos t - \sin t \end{bmatrix} = e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\Rightarrow$ General Solution:

$$\mathbf{x}(t) = X(t)\mathbf{c} + \mathbf{x}_p(t) = c_1 e^{2t}\begin{bmatrix} \sin t - \cos t \\ \cos t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -\cos t - \sin t \\ \sin t \end{bmatrix} + e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Ex. 26: Use e^{At} to the find the solution to the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, for

$$A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix},\ \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Answer: $T = 1,\ D = -2 \ \Rightarrow\ p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

$$A - 2I = \begin{bmatrix} 3 & 3 \\ * & * \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ A + I = \begin{bmatrix} 6 & 3 \\ * & * \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \text{F.S.S.: } \mathbf{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{x}_2(t) = e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \text{F.M.: } X(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ -e^{2t} & -2e^{-t} \end{bmatrix} \Rightarrow X(0) = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\Rightarrow \det(X(0)) = -1 \Rightarrow X(0)^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow e^{At} = X(t)X(0)^{-1} &= \begin{bmatrix} e^{2t} & e^{-t} \\ -e^{2t} & -2e^{-t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{2t} - e^{-t} & e^{2t} - e^{-t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{bmatrix} \end{aligned}$$

$$\Rightarrow \mathbf{x}(t) = e^{At}\mathbf{x}_0 = \begin{bmatrix} 2e^{2t} - e^{-t} & e^{2t} - e^{-t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^{2t} - 2e^{-t} \\ 4e^{-t} - 3e^{2t} \end{bmatrix}$$