9.8 (4.3): Higher (Second) Order Linear Equations

Form: $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = F(t)$ (1)

Equivalent 1st order system:

$x_1 = y,\ x_2 = y',\ \ldots,\ x_n = y^{(n-1)}$

$$\Rightarrow \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= -a_n(t)x_1 - \cdots - a_1(t)x_n \\ &\quad + F(t) \end{aligned}$$

or $\quad \mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t) \qquad (2)$

where $A(t) =$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t) \end{bmatrix}$$

$$\mathbf{f}(t) = [0, 0, \ldots, 0, F(t)]^T$$

$\mathbf{x}(t)$ is solution of (2) iff

$$\begin{aligned} \mathbf{x}(t) &= [x_1(t), x_2(t), \ldots, x_n(t)]^T \\ &= [y(t), y'(t), \ldots, y^{(n-1)}(t)]^T \end{aligned}$$

and $y(t)$ is solution of (1).

Initial Value Problem:

$\mathbf{x}(t_0) = \mathbf{x}_0 = [y_0, y_0', \ldots, y_0^{(n-1)}]^T \Leftrightarrow$

$y^{(j)}(t_0) = y_0^{(j)},\ 0 \le j \le n-1$ (3)

Thm: *(Existence/Uniqueness)*
Assume $a_1(t), \ldots, a_n(t), F(t)$ are continuous on an interval I and $t_0 \in I$. Then (1) with IC (3) has a unique solution on I for any values of $y_0, y_0', \ldots, y_0^{(n-1)}$.

Ex.: $my'' + \mu y' + ky = F(t)$, or

$$y'' + (\mu/m)y' + (k/m)y = F(t)/m$$

Equivalent system: $x_1 = y$, $x_2 = y' \Rightarrow$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -\mu/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F(t)/m \end{bmatrix}$$

Ex.: $y''' + a_1y'' + a_2y' + a_3y = F(t) \Rightarrow$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

Linear Independence, Wronskian, and General Solutions

Homogeneous ODE: $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = 0$ (4)

Def.: (*Linear Dependence and Independence of Functions*)
n functions $y_1(t), \ldots, y_n(t)$ are linearly dependent on an interval I if there are constants $c_1, \ldots, c_n$, not all zero,

s.t. $\quad c_1 y_1(t) + \ldots + c_n y_n(t) = 0$

for all $t \in I$. They are linearly independent on I if they are not linearly dependent.

Criterion for L.I.: Let $W(t) =$

$$\det \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}$$

be the *Wronskian* of $y_1(t), \ldots, y_n(t)$. If $W(t_0) \neq 0$ for some $t_0 \in I$, then $y_1(t), \ldots, y_n(t)$ are linearly independent on I.

Ex.: $y_1(t) = \cos t$, $y_2(t) = \sin t \Rightarrow$

$$W(t) = \begin{vmatrix} \cos t(t) & \sin t \\ -\sin t(t) & \cos t \end{vmatrix} = 1$$

Superposition Principle:
If $y_1(t), \ldots, y_k(t)$ are solutions of (4), then any linear combination

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_k y_k(t)$$

is also a solution.

Main Thm.:
Let $y_1(t), \ldots, y_n(t)$ be n solutions of (4) on an interval I on which $a_1(t), \ldots, a_n(t)$ are continuous.

(a) If $W(t_0) = 0$ for some $t_0 \in I$, then $y_1(t), \ldots, y_n(t)$ are linearly dependent on I.

(b) If $W(t_0) \neq 0$ for some $t_0 \in I$, then $W(t) \neq 0$ for all $t \in I$, and

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$$

is a general solution of (4).

(c) If $y_p(t)$ is a particular solution of (1) and $y(t)$ is a general solution of (4), then $y(t) + y_p(t)$ is a general solution of (1).

Homogeneous Equations with Constant Coefficients

Def.: A fundamental set of solutions (F.S.S.) for (4) is a set of linearly independent solutions $y_1(t), \dots, y_n(t)$.

Constant Coefficients Case:

($a_j =$ const)

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \quad (5)$$

Try solution $y(t) = e^{\lambda t} \Rightarrow$

$$p(\lambda) \equiv \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0 \quad (6)$$

Def.: (6) is called the characteristic equation of (5).

Associated Linear System:

$$\mathbf{x}' = A\mathbf{x} \quad (7)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

Thm.:

(a) The characteristic polynomial of A is $(-1)^n p(\lambda)$, i.e.: the eigenvalues of A are the roots of p.

(b1) If λ is a real root of p of multiplicity m, then (5) has the linearly independent solutions

$$e^{\lambda t},\ te^{\lambda t},\ \dots,\ t^{m-1}e^{\lambda t}$$

(b2) If $\lambda = \alpha + i\beta$ is a complex root of p of multiplicity m, then (5) has the linearly independent solutions

$$e^{\alpha t}\cos\beta t,\ te^{\alpha t}\cos\beta t, \dots, t^{m-1}e^{\alpha t}\cos\beta t$$
$$e^{\alpha t}\sin\beta t,\ te^{\alpha t}\sin\beta t, \dots, t^{m-1}e^{\alpha t}\sin\beta t$$

(c) The collection of functions obtained in (b) if λ runs through all roots, with only one root of a complex pair considered, is a F.S.S. for (5).

(d) If λ is any (real or complex) root of p, then $[1, \lambda, \dots, \lambda^{(n-1)}]^T$ is a basis of null$(A - \lambda I)$, i.e.: the geometric multiplicity of any eigenvalue of A is 1.

Examples for 2nd order equations

Ex.: $y'' - 3y' + 2y = 0$

IC: $y(0) = 2,\ y'(0) = 1$

$$p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

roots: $\lambda_1 = 2 \to y_1(t) = e^{2t}$
$\lambda_2 = 1 \to y_2(t) = e^{t}$

General solution: $y(t) = c_1 e^{2t} + c_2 e^{t}$

Match c_1, c_2 to IC:

$$\begin{array}{rcccl} y(0) & = & c_1 + c_2 & = & 2 \\ y'(0) & = & 2c_1 + c_2 & = & 1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\Rightarrow y(t) = -e^{2t} + 3e^{t}$$

Ex.: $y'' + 2y' + 2y = 0$

IC: $y(0) = 2,\ y'(0) = 3$

$$p(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda + 1)^2 + 1$$

roots: $\lambda = 1 + i,\ \overline{\lambda}$

$\Rightarrow y_1(t) = e^{-t}\cos t,\ y_2(t) = e^{-t}\sin t$

General solution:

$$y(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$$

Match c_1, c_2 to IC:

$$\begin{array}{rcccccc} y(0) & = & c_1 & = & 2 & & \\ y'(0) & = & -c_1 + c_2 & = & 3 & \Rightarrow & c_2 = 5 \end{array}$$

$$\Rightarrow y(t) = e^{-t}(2\cos t + 5\sin t)$$

Ex.: $y'' + 2y' + y = 0$

IC: $y(0) = 2,\ y'(0) = -1$

$$p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$$

double root $\lambda = -1 \Rightarrow$ F.S.S.: e^{-t}, te^{-t}

General solution: $y(t) = e^{-t}(c_1 + c_2 t)$

Match c_1, c_2 to IC:

$$\begin{array}{rcccccc} y(0) & = & c_1 & = & 2 & & \\ y'(0) & = & -c_1 + c_2 & = & -1 & \Rightarrow & c_2 = 1 \end{array}$$

$$\Rightarrow y(t) = e^{-t}(2 + t)$$

Examples for higher order equations

Ex.: $y''' + 6y'' + 11y' + 6y = 0$

$$\begin{aligned} p(\lambda) &= \lambda^3 + 6\lambda^2 + 11\lambda + 6 \\ &= (\lambda+1)(\lambda+2)(\lambda+3) \end{aligned}$$

$\Rightarrow$ roots: -1, -2, -3

$\Rightarrow$ F.S.S.: e^{-t}, e^{-2t}, e^{-3t}

$\Rightarrow$ General solution:

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}$$

Ex.: $y''' - y' = 0$

$$\begin{aligned} p(\lambda) &= \lambda^3 - \lambda = \lambda(\lambda^2 - 1) \\ &= \lambda(\lambda-1)(\lambda+1) \end{aligned}$$

$\Rightarrow$ roots: 0, 1, -1

$\Rightarrow$ F.S.S.: 1, e^t, e^{-t}

Ex.: $y'''' - 2y''' + 2y' - y = 0$

$$\begin{aligned} p(\lambda) &= \lambda^4 - 2\lambda^3 + 2\lambda - 1 \\ &= (\lambda-1)^3(\lambda+1) \end{aligned}$$

$\Rightarrow$ roots: 1 $(m = 3)$, -1 $(m = 1)$

$\Rightarrow$ F.S.S.: e^t, te^t, t^2e^t, e^{-t}

$\Rightarrow$ General solution:

$$y(t) = e^{-t}(c_1 + c_2 t + c_3 t^2) + c_4 e^{-t}$$

Ex.: $y'''' + 4y''' + 14y'' + 20y' + 25y = 0$

$$\begin{aligned} p(\lambda) &= \lambda^4 + 4\lambda^3 + 14\lambda^2 + 20\lambda + 25 \\ &= (\lambda^2 + 2\lambda + 5)^2 \end{aligned}$$

$$\Rightarrow \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 = 0$$

$\Rightarrow$ roots: $-1 \pm 2i$ $(m = 2)$

$\Rightarrow$ F.S.S.: $e^{-t}\cos 2t$, $te^{-t}\cos 2t$

$e^{-t}\sin 2t$, $te^{-t}\sin 2t$

Ex.: $y''' + y'' + y' + y = 0$ with IC

$y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$

(a) *Find general solution:*

$$\begin{aligned} p(\lambda) &= \lambda^3 + \lambda^2 + \lambda + 1 \\ &= (\lambda+1)(\lambda^2+1) \end{aligned}$$

$\Rightarrow$ roots: -1, i, $-i$

$\Rightarrow$ F.S.S.: e^{-t}, $\cos t$, $\sin t$

$$\Rightarrow \quad y(t) = c_1 e^{-t} + c_2 \cos t + c_3 \sin t$$

(b) *Match* c_1, c_2, c_3. Derivatives:

$$\begin{aligned} y'(t) &= -c_1 e^{-t} - c_2 \sin t + c_3 \cos t \\ y''(t) &= c_1 e^{-t} - c_2 \cos t - c_3 \sin t \end{aligned}$$

Equations for c_1, c_2, c_3:

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= -c_1 + c_3 = 0 \\ y''(0) &= c_1 - c_2 = 1 \end{aligned}\right\} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \\ c_3 = 1 \end{cases}$$

$$\Rightarrow \quad y(t) = e^{-t} + \sin t$$

Worked Out Examples from Exercises 9.8

Ex. 13: Consider $y''' + ay'' + by' + cy = 0$

(a) If $e^{\lambda t}$ is a solution provide details showing that $\lambda^3 + a\lambda^2 + b\lambda + c = 0$

Answer: $(e^{\lambda t})' = \lambda e^{\lambda t}$, $(e^{\lambda t})'' = (\lambda e^{\lambda t})' = \lambda^2 e^{\lambda t}$, $(e^{\lambda t})''' = (\lambda^2 e^{\lambda t})' = \lambda^3 e^{\lambda t}$.

$$\begin{aligned} \Rightarrow \quad (e^{\lambda t})''' + a(e^{\lambda t})'' + b(e^{\lambda t})' + ce^{\lambda t} &= \lambda^3 e^{\lambda t} + a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} \\ &= (\lambda^3 + a\lambda^2 + b\lambda + c)e^{\lambda t} = 0 \end{aligned}$$

Since $e^{\lambda t} \neq 0 \Rightarrow \lambda^3 + a\lambda^2 + b\lambda + c = 0$

(b) Write DE as 1st order system and compute characteristic polynomial

Answer: Set $x_1 = y$, $x_2 = y'$, $x_3 = y'' \Rightarrow x_1' = y' = x_2$, $x_2' = y'' = x_3$ and $x_3' = y''' = -ay'' - by' - cy = -ax_3 - bx_2 - cx_1 \Rightarrow$ system:

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -cx_1 - bx_2 - ax_3 \end{aligned}\right\} \Rightarrow \mathbf{x}' = A\mathbf{x} \text{ with } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix}$$

characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -c & -b & -a-\lambda \end{vmatrix} \\ &= (-\lambda)(-1)^{1+1}\begin{vmatrix} -\lambda & 1 \\ -b & -\lambda \end{vmatrix} + (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ -c & -a-\lambda \end{vmatrix} \\ &= -\lambda[(\lambda(\lambda + a) + b] - c = -(\lambda^3 + a\lambda^2 + b\lambda + c) \end{aligned}$$

Ex. 15: Find the general solution of $y''' - 3y'' - 4y' + 12y = 0$

$$p(\lambda) = \lambda^3 - 3\lambda^2 - 4\lambda + 12 = (\lambda^2 - 4)(\lambda - 3) = (\lambda - 2)(\lambda + 2)(\lambda - 3)$$

$$\Rightarrow \text{ F.S.S.: } e^{2t}, e^{-2t}, e^{3t} \Rightarrow y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 e^{3t}$$

Ex. 17: Find the general solution of $y^{(4)} - 13y'' + 36y = 0$

$$p(\lambda) = \lambda^4 - 13\lambda^2 + 36\lambda = (\lambda^2 - 9)(\lambda^2 - 4) = (\lambda - 3)(\lambda + 3)(\lambda - 2)(\lambda + 2)$$

$$\Rightarrow \text{ F.S.S.: } e^{3t}, e^{-3t}, e^{2t}, e^{-2t} \Rightarrow y(t) = c_1 e^{3t} + c_2 e^{-3t} + c_3 e^{2t} + c_4 e^{-2t}$$

Ex. 23: Find the general solution of $y''' + y'' - 8y' - 12y = 0$

$$p(\lambda) = \lambda^3 + \lambda^2 - 8\lambda - 12 = (\lambda + 2)^2(\lambda - 3)$$

$$\Rightarrow \text{ F.S.S.: } e^{-2t}, te^{-2t}, e^{3t} \Rightarrow y(t) = e^{-2t}(c_1 + c_2 t) + c_3 e^{3t}$$

Ex. 29: Find the general solution of $y''' - y'' + 2y = 0$

$$p(\lambda) = \lambda^3 - \lambda^2 + 2 = (\lambda + 1)(\lambda^2 - 2\lambda + 2) = (\lambda + 1)[(\lambda - 1)^2 + 1] \Rightarrow \text{roots } -1, 1 \pm i$$

$$\Rightarrow \text{ F.S.S.: } e^{-t}, e^t \cos t, e^t \sin t \Rightarrow y(t) = c_1 e^{-t} + e^t(c_2 \cos t + c_3 \sin t)$$

Ex. 33: Find the general solution of $y^{(6)} + 3y^{(4)} + 3y'' + y = 0$

$$p(\lambda) = \lambda^6 + 3\lambda^4 + 3\lambda^2 + 1 = (\lambda^2 + 1)^3 \Rightarrow \text{roots } \pm i \ (m = 3)$$

$$\Rightarrow \text{ F.S.S.: } \cos t, t\cos t, t^2 \cos t, \sin t, t \sin t, t^2 \sin t$$

$$\Rightarrow y(t) = (c_1 + c_2 t + c_3 t^2)\cos t + (c_4 + c_5 t + c_6 t^2)\sin t$$

Ex. 35: Find solution to IVP $y'' + 2y' + 5y = 0,\ y(0) = 2,\ y'(0) = 0$

$p(\lambda) = \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 \Rightarrow$ roots $-1 \pm 2i$

$$\Rightarrow \text{ F.S.S.: } e^{-t}\cos 2t, e^{-t}\sin 2t \ \Rightarrow\ y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$$

$$y'(t) = e^{-t}[(2c_2 - c_1)\cos 2t - (c_2 + 2c_1)\sin 2t] \Rightarrow \left\{ \begin{array}{c} y(0) = c_1 = 2 \\ y'(0) = -c_1 + 2c_2 = 0 \end{array} \right\}$$

$$\Rightarrow c_1 = 2,\ c_2 = 1 \Rightarrow y(t) = e^{-t}(2\cos 2t + \sin 2t)$$

Ex. 37: Find solution to IVP $y'' - 2y' + y = 0,\ y(0) = 1,\ y'(0) = 0$

$p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \Rightarrow$ F.S.S. e^t, te^t

$$\Rightarrow y(t) = e^t(c_1 + c_2 t) \Rightarrow y'(t) = e^t(c_1 + c_2 + c_2 t)$$

$$\text{IC: } \left\{ \begin{array}{c} y(0) = c_1 = 1 \\ y'(0) = c_1 + c_2 = 0 \end{array} \right\} \Rightarrow c_1 = 1,\ c_2 = -1 \Rightarrow y(t) = e^t(1 - t)$$

Ex. 39: Find solution to IVP

$$y''' - 7y'' + 11y' - 5y = 0,\ y(0) = -1,\ y'(0) = 1,\ y''(0) = 0$$

$p(\lambda) = \lambda^3 - 7\lambda^2 + 11\lambda - 5 = (\lambda - 1)^2(\lambda - 5) \Rightarrow$ F.S.S. $e^t,\ te^t,\ e^{5t}$

$$\Rightarrow y(t) = e^t(c_1 + c_2 t) + c_3 e^{5t} \Rightarrow y'(t) = e^t(c_1 + c_2 + c_2 t) + 5c_3 e^{5t}$$

$$\Rightarrow y''(t) = e^t(c_1 + 2c_2 + c_2 t) + 25 c_3 e^{5t}$$

$$\text{IC: } \left\{ \begin{array}{c} y(0) = c_1 + c_3 = -1 \\ y'(0) = c_1 + c_2 + 5c_3 = 1 \\ y''(0) = c_1 + 2c_2 + 25c_3 = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{ccr} c_1 & = & -13/16 \\ c_2 & = & 11/4 \\ c_3 & = & -3/16 \end{array} \right\}$$

$$\Rightarrow y(t) = (1/16)e^t(44t - 13) - (3/16)e^{5t}$$

Ex. 43: Find solution to IVP

$$y^{(4)} + 8y'' + 16y = 0,\ y(0) = 0,\ y'(0) = -1,\ y''(0) = 2,\ y'''(0) = 0$$

$$p(\lambda) = \lambda^4 + 16\lambda^2 + 8 = (\lambda^2 + 4)^2 \Rightarrow \text{roots } \pm 2i\ (m = 2)$$

$$\Rightarrow \text{ F.S.S.: } \cos 2t,\ t\cos 2t,\ \sin 2t,\ t\sin 2t$$

$$\Rightarrow\ y(t) = (c_1 + c_2 t)\cos 2t + (c_3 + c_4 t)\sin 2t$$

Computation of derivatives is cumbersome, so use Matlab's symbolic toolbox to find derivatives at $t = 0$ (only these are needed):

```
>> syms t c1 c2 c3 c4;y=(c1+c2*t)*cos(2*t)+(c3+c4*t)*sin(2*t);
>> y0=subs(y,t,0);yp=diff(y,t);yp0=subs(yp,t,0);ypp=diff(yp,t);ypp0=subs(ypp,t,0);
>> yppp=diff(ypp,t);yppp0=subs(yppp,t,0);[y0;yp0;ypp0;yppp0]
ans =
[          c1]
[     c2+2*c3]
[  -4*c1+4*c4]
[ -12*c2-8*c3]
```

Thus the equations for matching c_1, c_2, c_3, c_4 are

$$\left\{\begin{array}{c} y(0) = c_1 = 0 \\ y'(0) = c_2 + 2c_3 = -1 \\ y''(0) = -4c_1 + 4c_4 = 2 \\ y'''(0) = -12c_2 - 8c_3 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{ccr} c_1 & = & 0 \\ c_2 & = & 1/2 \\ c_3 & = & -3/4 \\ c_4 & = & 1/2 \end{array}\right\}$$

$$\Rightarrow\ y(t) = (t/2)\cos 2t + (t/2 - 3/4)\sin 2t$$