7.4: Homogeneous and Inhomogeneous Systems

Homogeneous Systems

Form: $A\mathbf{x} = \mathbf{0}$ (1)

- (1) has always solution $\mathbf{x} = \mathbf{0}$
- Any solution $\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$ is called *nontrivial*
- Nontrivial solutions exist if $REF(A)$ or $RREF(A)$ has a free column
- Let p: ♯ pivots
  f: ♯ free variables
  Note: $p \leq m$, $n = p + f$
- If $m < n$ (fewer equations than unknowns) $\Rightarrow f > 0 \Rightarrow$ (1) has nontrivial solutions

Ex.: Consider $A\mathbf{x} = \mathbf{0}$ for

$$A = \begin{bmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -5 \end{bmatrix}$$

Note: Since $\mathbf{b} = \mathbf{0}$ there is no need to augment A

Use Matlab's *rref* command $\Rightarrow$

$$RREF(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

- x_1, x_2: pivot variables
- $x_3 = t$: free variable

Equations: $\begin{cases} x_1 - t & = & 0 \\ x_2 - t/2 & = & 0 \end{cases}$

Solution:

$$\mathbf{x} = \begin{bmatrix} t \\ t/2 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

Ex.: $A\mathbf{x} = \mathbf{0}$ for

$$A = \begin{bmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{bmatrix}$$

Use Matlab $\Rightarrow$

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Equations:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

No free variables

$\Rightarrow$ only trivial solution $\mathbf{x} = 0$

Ex.: $A\mathbf{x} = \mathbf{0}$ for

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Apply successively

$$R3(1, -1),\ R1(2, 1, -1),\ R1(1, 2, 1)$$

$$\Rightarrow \quad RREF(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

- x_1, x_2: pivot variables
- $x_3 = t$: free variable

Equations: $\begin{cases} x_1 + 2t = 0 \\ x_2 + 3t = 0 \end{cases}$

Solution:

$$\mathbf{x} = \begin{bmatrix} -2t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Nullspaces and Subspaces

Nullspaces

Def. A: $m \times n$–matrix

$$\text{null}(A) \stackrel{def}{=} \{\mathbf{x} \in \mathbf{R}^n \mid A\mathbf{x} = \mathbf{0}\} = \text{nullspace of } A$$

Properties:

1. If $\mathbf{x}, \mathbf{y}$ are in null(A) then $\mathbf{x} + \mathbf{y}$ is in null(A)
2. If $\mathbf{x}$ is in null(A) then $a\mathbf{x}$ is in null(A) for any number a

Proof:

1. If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ then
$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$
2. If $A\mathbf{x} = \mathbf{0}$ then
$A(a\mathbf{x}) = aA\mathbf{x} = a\mathbf{0} = \mathbf{0}$

Subspaces (see 7.5)

Def. A set V in $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$ if

1. If $\mathbf{x}, \mathbf{y} \in V$ then $\mathbf{x} + \mathbf{y} \in V$
2. If $\mathbf{x} \in V$ then $a\mathbf{x} \in V$ for any number a

Properties: any subspace V

- contains $\mathbf{0}$
- is nullspace of a matrix

Examples:

- a line through $\mathbf{0}$ in $\mathbf{R}^2$ is a subspace
- a line in $\mathbf{R}^2$ that does *not* cross $\mathbf{0}$ is *not* a subspace
- trivial subspaces: $\mathbf{R}^n$ and $\mathbf{0}$

Find nullspace

Transform A to REF or RREF

- If all variables are pivot $\Rightarrow$ null$(A) = \{\mathbf{0}\}$
- If there are free variables, solve for pivot variables in terms of free parameters $\Rightarrow$ nontrivial nullspace

Ex.: Find nullspace of

$$A = \begin{bmatrix} 0 & 1 & 3 & 2 & 2 \\ 1 & 2 & 3 & 5 & 7 \\ 2 & 4 & 6 & 9 & 12 \end{bmatrix}$$

$$A \xrightarrow{R2(1,2)} \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 1 & 3 & 2 & 2 \\ 2 & 4 & 6 & 9 & 12 \end{bmatrix}$$

$$\xrightarrow{R1(3,1,-2)} \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R3(3,-1)} \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R1(1,3,-5),R1(2,3,-2)} \begin{bmatrix} 1 & 2 & 3 & 0 & 12 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R1(1,2,-2)} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Free: $x_3 = s$, $x_5 = t$

Pivots:

$$\begin{aligned} x_1 &= 3x_3 - 4x_5 &= 3s - 4t \\ x_2 &= -3x_3 - 4x_5 &= -3s - 4t \\ x_4 &= x_5 &= t \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathbf{x} &= [3s - 4t, -3s - 4t, s, t, t]^T \\ &= s\mathbf{v}_1 + t\mathbf{v}_2 \end{aligned}$$

where $\mathbf{v}_1 = [3, -3, 1, 0, 0]^T$
$\mathbf{v}_2 = [-4, -4, 0, 1, 1]^T$

$\Rightarrow$ null$(A) = \{\mathbf{x} = s\mathbf{v}_1 + t\mathbf{v}_2 \mid s, t \in \mathbf{R}\}$

Solution Structure of Inhomogeneous Systems

Let A: $m \times n$, $\mathbf{b} \in \mathbf{R}^m$. Consider

$$\text{NHE:} \quad A\mathbf{x} = \mathbf{b} \qquad (2)$$

$$\text{HE:} \quad A\mathbf{x} = \mathbf{0} \qquad (3)$$

Thm: Let $\mathbf{x}_p$ be a particular solution of (2).

1. If $\mathbf{x}_h$ satisfies (3), then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ satisfies (2)
2. If $\mathbf{x}$ satisfies (2), then $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$ satisfies (3)

Consequence:

Given a particular solution $\mathbf{x}_p$, the full solution set S of (2) is

$$S = \{\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h \,|\, A\mathbf{x}_h = \mathbf{0}\}$$

• If $\mathbf{b} \neq \mathbf{0}$, S is *not* a subspace

Ex.: $A = [1, -1]$, $\mathbf{b} = 1$

$$\text{NHE:} \quad x - y = 1 \qquad (4)$$

$$\text{HE:} \quad x - y = 0 \qquad (5)$$

(4) $\Rightarrow y = t,\ x = 1 + t \Rightarrow \mathbf{x} = \mathbf{x}_p + t\mathbf{x}_h$

where $\mathbf{x}_p = [1, 0]^T \ : \ [1, -1]\mathbf{x}_p = 1$

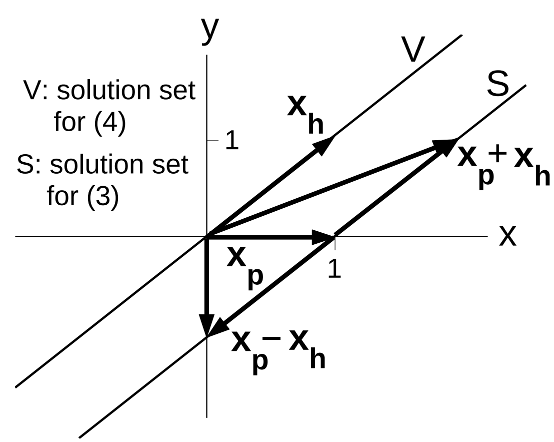
$\mathbf{x}_h = [1, 1]^T \ : \ [1, -1]\mathbf{x}_h = 0$

Solution set V of (5) (subspace):

$$V = \text{null}([1, -1]) = \{\mathbf{x} = t\mathbf{x}_h \,|\, t \in \mathbf{R}\}$$

Solution set S of (4) (not subspace):

$$S = \{\mathbf{x} = \mathbf{x}_p + t\mathbf{x}_h \,|\, t \in \mathbf{R}\}$$



Find Particular Solution

- Transform $M = [A, \mathbf{b}]$ to REF or RREF
- If system is consistent (last column of $REF(M)$ or $RREF(M)$ is not pivot), set all free variables zero and solve for pivot variables

Ex.: Find particular solution to $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 0 & 1 & 3 & 2 & 2 \\ 1 & 2 & 3 & 5 & 7 \\ 2 & 4 & 6 & 9 & 12 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$$

$$\Rightarrow \; M = \begin{bmatrix} 0 & 1 & 3 & 2 & 2 & 1 \\ 1 & 2 & 3 & 5 & 7 & 8 \\ 2 & 4 & 6 & 9 & 12 & 2 \end{bmatrix}$$

Apply same row operations as on p.4 to $M \Rightarrow$

$$RREF(M) = \begin{bmatrix} 1 & 0 & -3 & 0 & 4 & -8 \\ 0 & 1 & 3 & 0 & 4 & -27 \\ 0 & 0 & 0 & 1 & -1 & 14 \end{bmatrix}$$

Set free variables zero:

$$\begin{aligned} x_3 &= 0 \\ x_5 &= 0 \end{aligned}$$

Solve for pivot variables:

$$\begin{aligned} x_1 &= -8 \\ x_2 &= -27 \\ x_4 &= 14 \end{aligned}$$

$\Rightarrow$ particular solution:

$$\mathbf{x}_p = [-8, -27, 0, 14, 0]^T$$

Full solution set:

$$S = \{\mathbf{x} = \mathbf{x}_p + s\mathbf{v}_1 + t\mathbf{v}_2 \,|\, s, t \in \mathbf{R}\},$$

with $\mathbf{v}_1, \mathbf{v}_2$ given on p.4