Chapter 6: Numerical Methods

6.1 Euler Method

Basic Idea

- ODE: $y' = f(t,y)$
- Assume $y(t)$ is known
- For small h approximate

$$\frac{y(t+h)-y(t)}{h} \approx y'(t) = f(t,y(t))$$

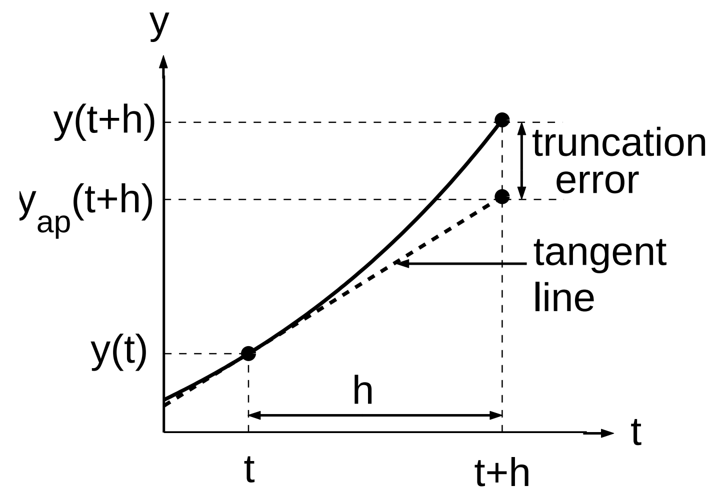
$$\Rightarrow \quad y(t+h) \approx y_{ap}(t+h)$$

where

$$y_{ap}(t+h) = y(t) + h\,f(t,y(t))$$

- **Truncation Error:**

$$|y(t+h) - y_{ap}(t+h)|$$



Iteration Scheme

IVP: $y' = f(t,y)$, $y(t_0) = y_0$

Approximate $y(t_k) \approx y_k$ at t_k:

$$y_1 = y_0 + h\,f(t_0,y_0), \quad t_1 = t_0 + h$$
$$y_2 = y_1 + h\,f(t_1,y_1), \quad t_2 = t_1 + h$$
$$\vdots$$
$$y_{k+1} = y_k + h\,f(t_k,y_k)$$
$$t_{k+1} = t_k + h$$

Ex: Approximate the solution to

$$y' = y, \quad y(0) = 1$$

in $0 \le t \le 1$. Start: $t_0 = 0$, $y_0 = 1$

$\mathbf{h = 1}$

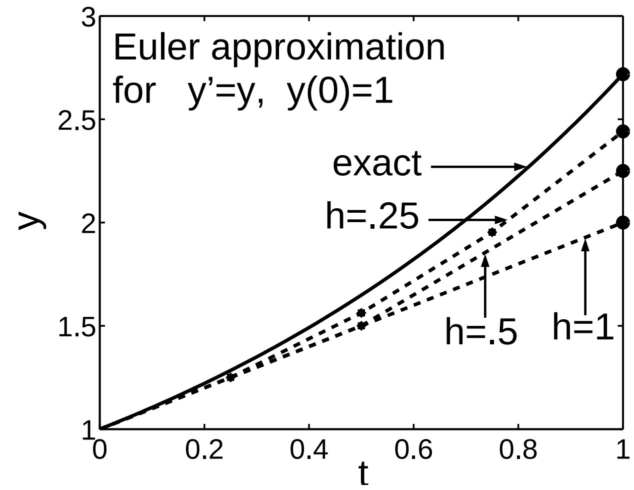
$$\begin{aligned} y_1 &= y_0 + h\,f(0,1) = 1 + 1 \cdot 1 = 2 \\ t_1 &= t_0 + h = 0 + 1 = 1 \end{aligned}$$

$\mathbf{h = 0.5}$

$$\begin{aligned} y_1 &= 1 + 0.5 \cdot 1 = 1.5 \\ t_1 &= 0 + 0.5 = 0.5 \\ y_2 &= 1.5 + 0.5 \cdot 1.5 = 2.25 \\ t_2 &= 0.5 + 0.5 = 1 \end{aligned}$$

$\mathbf{h = 0.25}$

$$\begin{aligned} y_1 &= 1 + 0.25 \cdot 1 = 1.25 \\ t_1 &= 0 + 0.25 = 0.25 \\ y_2 &= 1.25 + 0.25 \cdot 1.25 = 1.5625 \\ t_2 &= 0.25 + 0.25 = 0.5 \\ y_3 &= 1.5625 + 0.25 \cdot 1.5625 \\ &= 1.953125 \\ t_3 &= 0.5 + 0.25 = 0.75 \\ y_4 &= 1.953125 + 0.25 \cdot 1.953125 \\ &= 2.44140625 \\ t_4 &= 0.75 + 0.25 = 1 \end{aligned}$$



Ex: Approximate the solution to

$$y' = t - y, \quad y(0) = 0.5$$

in $0 \le t \le 1$ using $h = 0.25$

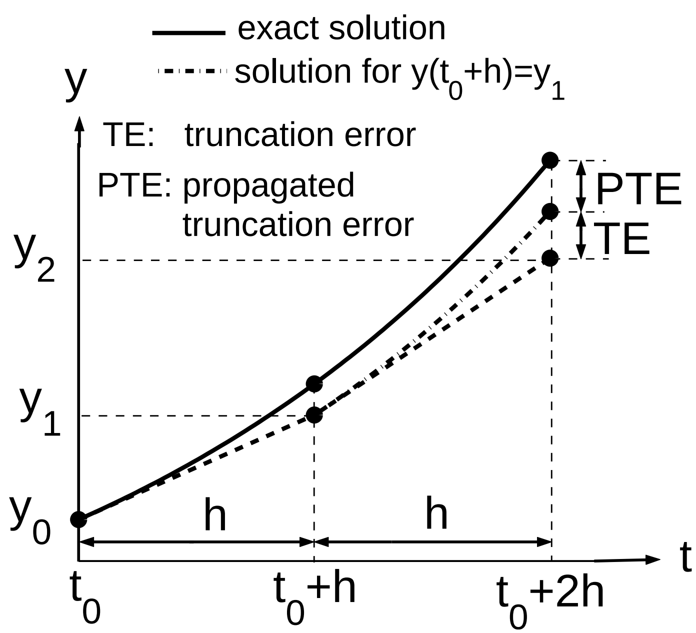
Start: $y_0 = 0.5$, $t_0 = 0$

$$\begin{aligned} y_1 &= 0.5 + 0.25 \cdot (0 - 0.5) = 0.375 \\ t_1 &= 0 + 0.25 = 0.25 \\ y_2 &= 0.375 + 0.25 \cdot (0.25 - 0.375) \\ &= 0.3438 \\ t_2 &= 0.25 + 0.25 = 0.5 \\ y_3 &= 0.3438 + 0.25 \cdot (0.5 - 0.3438) \\ &= 0.3828 \\ t_3 &= 0.5 + 0.25 = 0.75 \\ y_4 &= 0.3828 + 0.25 \cdot (0.75 - 0.3828) \\ &= 0.4746 \\ t_4 &= 0.75 + 0.25 = 1 \end{aligned}$$

Errors

Three error sources:

- Truncation error at each Euler step
- Propagated (accumulated) truncation error
- Roundoff error (not controlable)



Ex.: $y' = t - y, \quad y(0) = .5$
Approximate $y(1)$ for stepsizes

$$h = 1/m, \quad m = 1, 2, 4, 8, 16, 32$$

Exact Value: $y(1) = 0.5518$

Error: $E(h) = |y(1) - y_m|$

h	y_m	$E(h)$
1	0	0.5518
1/2	0.375	0.1768
1/4	0.4746	0.0772
1/8	0.5154	0.0364
1/16	0.5341	0.0177
1/32	0.5431	0.0087

$$E(h/2) \approx E(h)/2 \;\Rightarrow\; E(h) \approx C\,h$$

Theorem: There $\exists\ C > 0$ s.t.

$$E(h) \leq C\,h$$

(Euler method is first order method)

Worked out Examples from Exercises

Ex. 1: $y' = ty$, $y(0) = 1$.

Compute five Euler-iterates for $h = 0.1$.
Arrange computation and results in a table.

k	t_k	y_k	$f(t_k, y_k) = t_k y_k$	h	$f(t_k, z_k)h$
0	0	1	0	0.1	0
1	0.1	1	0.1000	0.1	0.0100
2	0.2	1.0100	0.2020	0.1	0.0202
3	0.3	1.0302	0.3091	0.1	0.0309
4	0.4	1.0611	0.4244	0.1	0.0424
5	0.5	1.1036	0.5518	0.1	0.0552

Ex. 7: $y' + 2xy = x$, $y(0) = 8$

(i) Compute Euler-approximations in $0 \le x \le 1$ for $h = 0.2$, $h = 0.1$, $h = 0.05$.
(ii) Find exact solution
(iii) Plot exact solution as curve and Euler approximations as points.

(i) In Matlab, Euler approximation for $h = 0.2$ is computed and stored in arrays *x0_2*, *y0_2* via

```
h=0.2;
m=1/h;x=0;y=8;
xv=x;yv=y;
for k=1:m
    f=-2*x*y+x;
    y=y+h*f;yv=[yv y];
    x=x+h;xv=[xv x];
end
x0_2=xv;y0_2=yv;
```

Analogously for $h = 0.1$ and $h = 0.05$ (arrays *x0_1*, *y0_1* and *x0_05*, *y0_05*).

(ii) Variation of Parameter:

$$y_h' = -2xy \quad \Rightarrow$$

$$y_h(x) = \exp\Big(\int_0^x (-2x)dx\Big) = e^{-x^2}$$

$$\begin{aligned} y(x) &= y_h(x)\Big(8 + \int_0^x [f(\xi)/y_h(\xi)]d\xi\Big) \\ &= 8e^{-x^2} + e^{-x^2}\int_0^x \xi e^{\xi^2} d\xi \\ &= 8e^{-x^2} + e^{-x^2}(e^{x^2} - 1)/2 \\ &= (15/2)e^{-x^2} + 1/2 \end{aligned}$$

(iii) Matlab plot commands:

```
x=linspace(0,1,100);
y=1/2+15/2*exp(-x.^2);
plot(x0_2,y0_2,'ko',x0_1,y0_1,'k*',...
     x0_05,y0_05,'k+',x,y,'k'),
xlabel('x'),ylabel('y')
axis([0 1 3.5 8])
```

Plot for Ex. 7