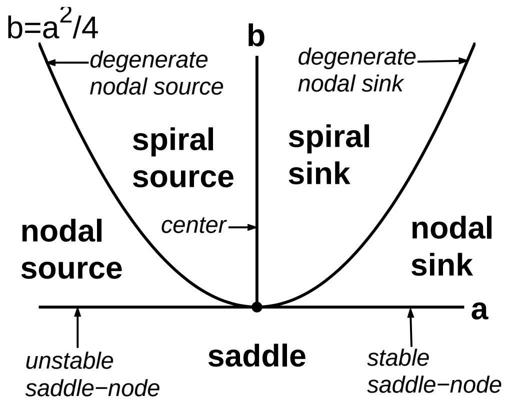
4.2: Phase Plane Portraits; 4.4: Free Harmonic Motion

$$y'' + ay' + by = 0 \qquad (1)$$
$$p(\lambda) = \lambda^2 + a\lambda + b = 0$$

Associated linear system:

$$x_1 = y,\ x_2 = y' \ \Rightarrow$$
$$\mathbf{x}' = A\mathbf{x},\ A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \qquad (2)$$
$$T = -a,\ D = b$$
$$\det(A - \lambda I) = p(\lambda)$$

- Phase plane portrait for DE (1) = phase plane portrait of (2)
- Classification as in Section 9.3 with $T = -a$, $D = b$

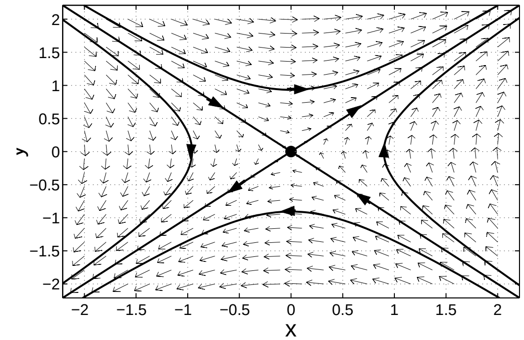


Ex.: $y'' - y = 0$ $(a = 0, b = -1)$

$$p(\lambda) = \lambda^2 - 1 \Rightarrow \lambda = \pm 1 \text{ (saddle)}$$

General solution: $y(t) = c_1 e^t + c_2 e^{-t}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{cases} \lambda_1 = 1 & \leftrightarrow \mathbf{v}_1 = [1, 1]^T \\ \lambda_2 = -1 & \leftrightarrow \mathbf{v}_2 = [-1, 1]^T \end{cases}$$

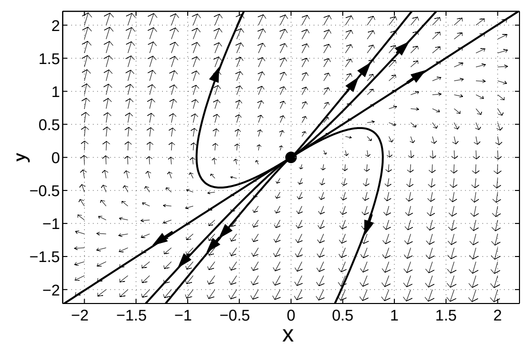


Ex.: $y'' - 3y' + 2y = 0$

$$p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

$\Rightarrow$ source: $y(t) = c_1 e^t + c_2 e^{2t}$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \rightarrow \begin{cases} \lambda_1 = 1 \leftrightarrow \mathbf{v}_1 = [1, 1]^T \\ \lambda_2 = 2 \leftrightarrow \mathbf{v}_2 = [1, 2]^T \end{cases}$$

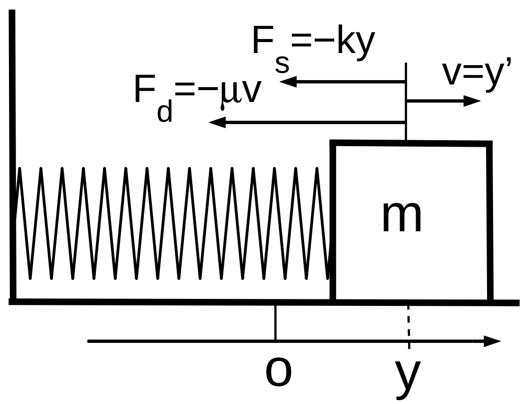


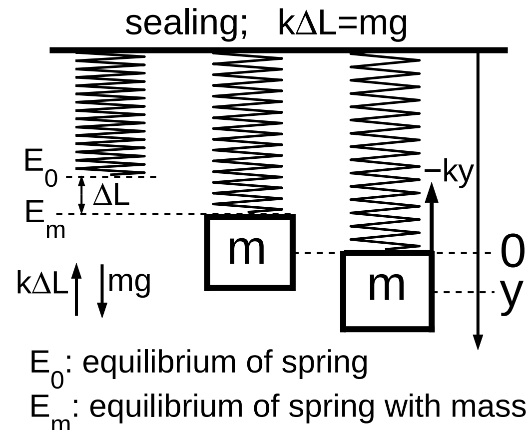
4.4: Free Harmonic Motion

Mass–spring system:

$$my'' + \mu y' + ky = 0$$

- m: mass (kg)
- μ: damping constant (kg/s)
- k: spring constant (kg/s^2)
- y: deviation of mass position from equilibrium position (m)
- y': velocity (m/s)

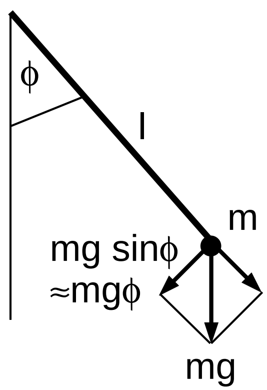




Pendulum for small ϕ:

$$\phi'' + (\mu/m)\phi' + (g/l)\phi = 0$$

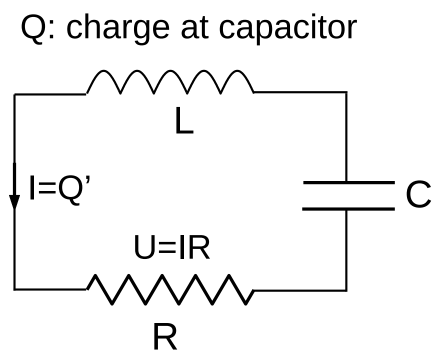
- ϕ: angle (no unit)
- g: $9.8\,m/s^2$
- l: length (m)



RLC–circuit:

$$LQ'' + RQ' + Q/C = 0$$

- Q: charge (C)
- I: current Q' (A)
- L: inductivity (H)
- R: resistor (Ω)
- C: capacity (F)



Classification of Harmonic Motion for Mass Spring Systems

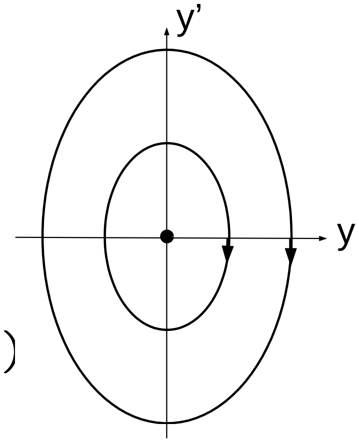
$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0 \;\Big|\; \lambda_{1,2} = -\frac{\mu}{2m} \pm \frac{1}{2m}\sqrt{\mu^2 - 4km} \;\Big|\; m, k > 0 \;\Big|\; \mu \geq 0$$

Undamped Case: $\mu = 0$

$$\lambda = \pm i\omega_0$$

$$\omega_0 = \sqrt{k/m}$$

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$



- oscillation
- phase portrait: center
- clockwise direction of rotation

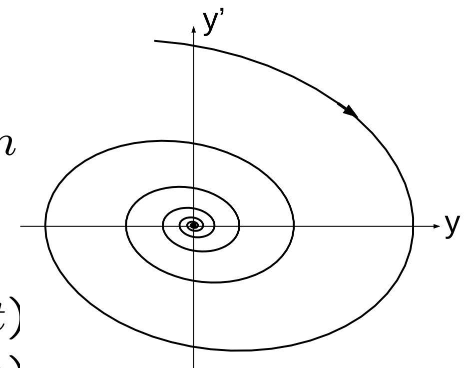
Underdamped Case: $0 < \mu^2 < 4km$

$$\lambda_{1,2} = -\alpha \pm i\omega$$

$$\alpha = \mu/2m$$

$$\omega = \sqrt{4km - \mu^2}/(2m) = \sqrt{\omega_0^2 - \mu^2/4m^2}$$

$$y(t) = e^{-\alpha t}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$$



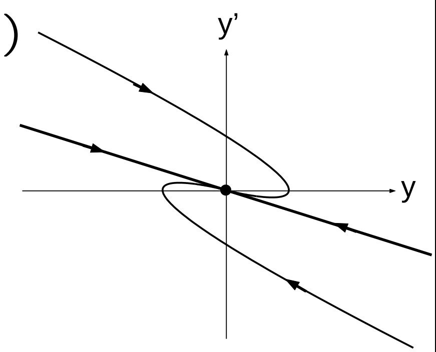
- damped oscillation
- phase portrait: spiral sink
- clockwise direction of rotation

Critically Damped Case: $\mu^2 = 4km$

$$\lambda_1 = \lambda_2 = -\mu/(2m)$$

$$y(t) = e^{\lambda_1 t}(c_1 + c_2 t)$$

- phase portrait: degenerate nodal sink

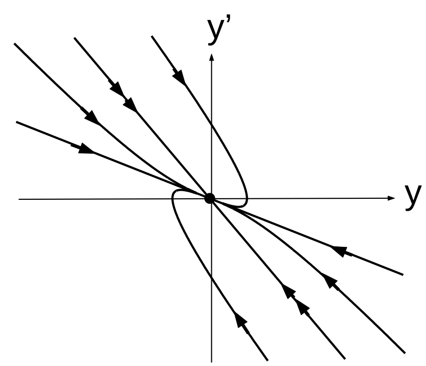


Overdamped Case: $\mu^2 > 4km$

$$\lambda_1 < \lambda_2 < 0$$

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

- phase portrait: nodal sink
- both eigenlines: negative slopes



Qualitative Features of Harmonic Motion

Undamped Case:

$$\begin{aligned} y(t) &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ &= A[d_1 \cos(\omega_0 t) + d_2 \sin(\omega_0 t)] \end{aligned}$$

where $\begin{cases} A = \sqrt{c_1^2 + c_2^2} \\ d_1 = c_1/A,\ d_2 = c_2/A \end{cases}$

Since $d_1^2 + d_2^2 = 1$ we can define ϕ by

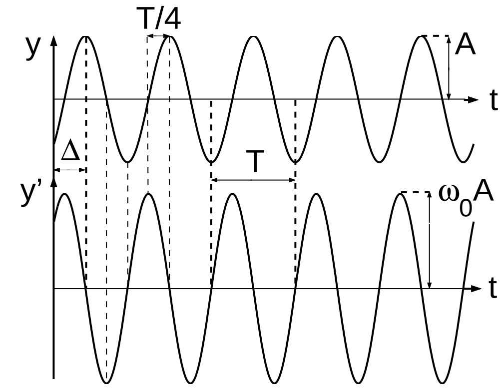
$$d_1 = \cos\phi, \quad d_2 = \sin\phi$$
$$\Rightarrow \quad d_2/d_1 = c_2/c_1 = \tan\phi$$

$$\begin{aligned} y(t) &= A[\cos\phi \cos(\omega_0 t) + \sin\phi \sin(\omega_0 t)] \\ &= A\cos(\omega_0 t - \phi) \\ y'(t) &= -\omega_0 A \sin(\omega_0 t - \phi) \end{aligned}$$

$\mathbf{A}$: **amplitude**

ϕ: **phase angle**, choose $-\pi < \phi \le \pi$

$$\phi = \begin{cases} \arctan(c_2/c_1) & \text{if } c_1 > 0 \\ \arctan(c_2/c_1) + \pi & \text{if } c_1 < 0, c_2 \ge 0 \\ \arctan(c_2/c_1) - \pi & \text{if } c_1 < 0, c_2 < 0 \\ \pi/2 & \text{if } c_1 = 0, c_2 > 0 \\ -\pi/2 & \text{if } c_1 = 0, c_2 < 0 \end{cases}$$

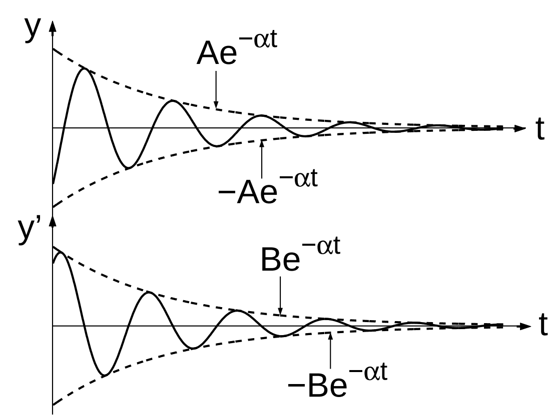


$\Delta = (\phi/2\pi)T, \ T = 2\pi/\omega_0$

Underdamped Case:

$$\begin{aligned} y(t) &= e^{-\alpha t}[c_1 \cos(\omega t) + c_2 \sin(\omega t)] \\ &= e^{-\alpha t} A \cos(\omega t - \phi) \\ y'(t) &= e^{-\alpha t}[(\omega c_2 - \alpha c_1)\cos(\omega t) \\ &\qquad -(\omega c_1 + \alpha c_2)\sin(\omega t)] \\ &= e^{-\alpha t} B \cos(\omega t - \psi) \end{aligned}$$

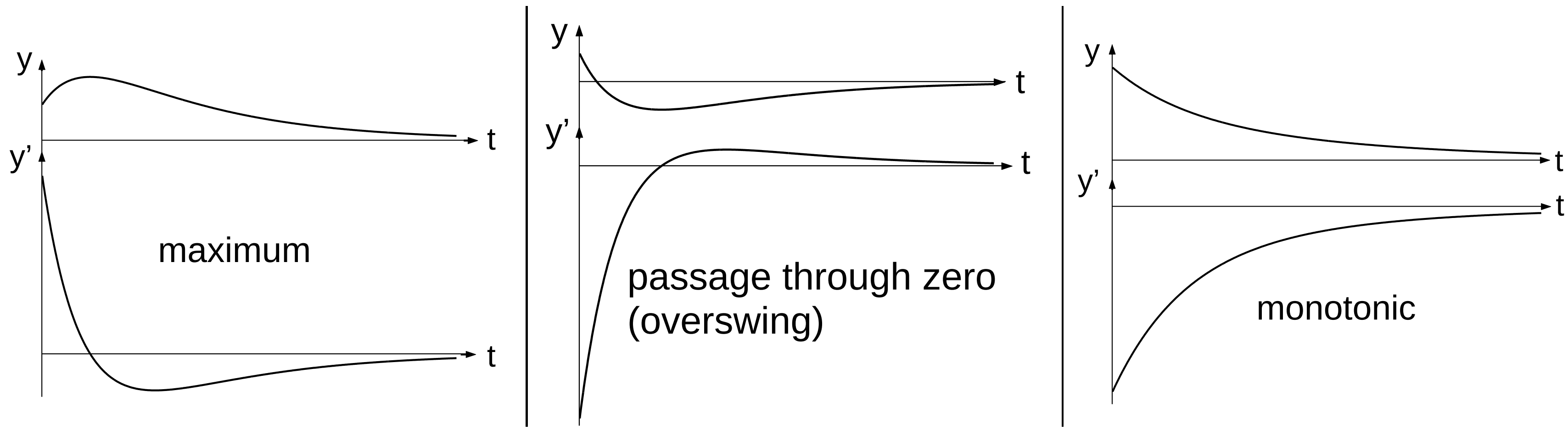
$\pm A e^{-\alpha t}, \pm B e^{-\alpha t}$: envelopes of damped oscillations



Critically and Overdamped Cases: (Details $\rightarrow$ homework)

- If $y(0)$ and $y'(0)$ have equal signs, then
  - $y(t)$ attains maximum or minimum
  - $y'(t)$ crosses zero

- If $y(0)$ and $y'(0)$ have opposite signs, then $y(t)$
  - crosses zero if $|y'(0)/y(0)|$ is large
  - is monotonic if $|y'(0)/y(0)|$ is small



Ex. 4.4.11: Given an undamped mass-spring system with $m = 0.2\,kg$, $k = 5\,kg/s^2$, $y(0) = 0.5\,m$, $y'(0) = 0$, find amplitude, frequency, phase of motion.

Natural frequency: $\omega_0 = \sqrt{5/0.2} = 5/s \Rightarrow$

$$y(t) = c_1 \cos 5t + c_2 \sin 5t, \quad y'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t$$

$$\text{IC: } y(0) = c_1 = 0.5,\ y'(0) = 5c_2 = 0 \ \Rightarrow\ y(t) = 0.5 \cos 5t$$

$$\Rightarrow \ \text{amplitude: } A = 0.5\,m, \quad \text{phase: } \phi = 0$$

Ex. 4.4.22: A mass-spring system with $m = 0.1\,kg$, $k = 9.8\,kg/s^2$ is placed in a viscous medium with friction force $0.3\,N$ if $v = 0.2\,m/s$. Initial data: $y(0) = 0.1\,m$, $y'(0) = 0$. Find amplitude, frequency, and phase of motion.

Friction coefficient: $F_d = \mu v \Rightarrow 0.3 = \mu 0.2 \Rightarrow \mu = 1.5\,kg/s$.

$$\text{ODE: } \ 0.1y'' + 1.5y' + 9.8y = 0 \ \Rightarrow\ y'' + 15y' + 98y = 0 \ \Rightarrow$$

$$p(\lambda) = \lambda^2 + 15\lambda + 98 = (\lambda + 7.5)^2 + 41.75 \ \Rightarrow\ \lambda = -7.5 \pm i\sqrt{41.75} \approx -7.5 \pm 6.461i$$

$\Rightarrow$ Damped motion with frequency $\omega \approx 6.461\,/s$ of harmonic part $\Rightarrow$

$$y(t) = e^{-7.5t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

$$y'(t) = e^{-7.5t}[(\omega c_2 - 7.5c_1) \cos \omega t - (\omega c_1 + 7.5c_2) \sin \omega t]$$

$$\text{IC: } \ y(0) = c_1 = 0.1, \ \ y'(0) = \omega c_2 - 7.5c_1 = 0 \ \ \Rightarrow \ \ c_2 = 7.5c_1/\omega \approx 0.116$$

$$\Rightarrow \ y(t) = e^{-7.5t}(0.1 \cos \omega t + 0.116 \sin \omega t)$$

Amplitude of harmonic part: $A_0 \approx \sqrt{0.1^2 + 0.116^2} \approx 0.153$. Since $c_1, c_2 > 0$ $\Rightarrow$ phase angle $\phi = \arctan(c_2/c_1) \approx \arctan(1.16) \approx 0.859$

$$\Rightarrow \ y(t) = 0.153e^{-7.5t} \cos(6.461t - 0.859)$$

$\Rightarrow$ amplitude: $A(t) = 0.153e^{-7.5t}\,m$,
frequency: $\omega = 6.461/s$, phase: $\phi = 0.859$